ECE 409 - BASEBAND TRANSMISSION - INVESTIGATION 8
INTRODUCTION TO MATCHED FILTERS

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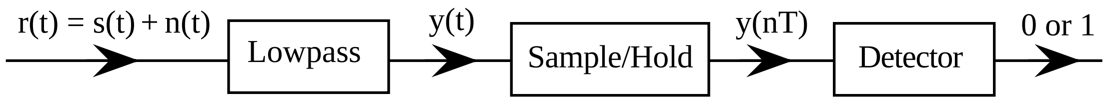
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

Up to now *center point detectors* as follows



to decide which received signals are 1's and which 0's by comparing sample values to corresponding threshold values. The objective of this Investigation is to introduce matched filters that reduce the affect of AWGN by taking advantage of the fact that even though the noise may be large at any given time its average over time T is usually close to zero.

1. The objective of this and the next several problems is to find an equation for the impulse response h(t) of the **matched filter** as follows

a filter that we are going to derive to minimize the affect of the AWGN channel noise $n(t)$ at the time nT_s when y(t) is being sampled. Note that we refer to such filters as *matched filters* because they depend on and are therefore *matched* to the input

a. Show that $y(t) = h(t) \quad s(t) + h(t) \quad n(t)$ where h(t) is the impulse response of the matched filter

b. From the result in part (a) we can write y(t) as follows

$$y(t) = s_o(t) + n_o(t)$$

where $s_o(t) = h(t) \quad s(t)$ and $w_o(t) = h(t) \quad n(t)$. Our goal now becomes to find the impulse response h(t) of the matched filter to maximize

$$(SNR)_o = \frac{\left|s_o(T_s)\right|^2}{E\left[n_o^2(t)\right]}$$

Describe in words what is being maximized.

c. To find $h(t)$ that maximizes $(SNR)_o$ we first need to express $\left|s_o(T_s)\right|^2$ and $E\left[n_o^2(t)\right]$ in terms of $h(t)$. To do this we first express $s_o(T_s)$ and $E\left[n_o^2(t)\right]$ in terms of the transfer function $G(jf)$ of the matched filter. First explain where the following expression for $s_o(t)$

$$s_o(t) = \int_{-\infty}^{\infty} S_o(f) e^{j2\pi ft} df = \int_{-\infty}^{\infty} G(jf) S(f) e^{j2\pi ft} df$$

and therefore where the following expression for $|s_o(T_s)|^2$ comes from

$$|s_o(T_s)|^2 = \left|\int_{-\infty}^{\infty} G(jf)S(f)e^{j2\pi fT_s}\,df\right|^2$$

where $G(jf)$ is the transfer function of the matched filter.

d. Make use of the following results relating the autocorrelation $R_n(\tau)$ and the noise spectral densities $S_n(f)$ and $S_{n_o}(f)$

$$S_{n_o}(f) = S_n(f)|G(jf)|^2 = \frac{N_o}{2}|G(jf)|^2$$

$$R_{n_o}(\tau) = E\left[n_o(t)n_o(t+\tau)\right] = F^{-1}\left[S_{n_o}(f)\right] = \int_{-\infty}^{\infty} S_{n_o}(f)e^{j2\pi f\tau}\,df$$

with $S_n(f) = N_o/2$ equal to the power spectral density of the white noise to show that

$$E\left[n_o^2(t)\right] = \frac{N_o}{2}\int_{-\infty}^{\infty}|G(jf)|^2\,df$$

2. The next step in this development is to find an upper bound on $(SNR)_o$. To do this we make use of Schwarz's Inequality as follows

$$\left|\int_{-\infty}^{\infty} f(x)g(x)dx\right|^2 \le \int_{-\infty}^{\infty}|f(x)|^2\,df\int_{-\infty}^{\infty}|g(x)|^2\,dx$$

a. Describe in words what Schwarz's Inequality says
b. Verify Schwarz's Inequality for $f(x) = e^{-x}u(x)$ and $g(x) = e^{-2x}u(x)$ where u(x) is the unit step function
c. Show that equality will hold if $f(x) = kg\ \ (x)$

3. The objective of this Problem is to make use of the results of Problems (1) and (2) to find an upper bound on the $(SNR)_o$ as follows

$$(SNR)_o = \frac{|s_o(T_s)|^2}{E\left[n_o^2(t)\right]} = \frac{\left|\int_{-\infty}^{\infty} G(jf)S(f)e^{j2\pi ft}df\right|^2}{\frac{N_o}{2}\int_{-\infty}^{\infty}|G(jf)|^2\,df}$$

To find this upper bound we make use of Schwarz's inequality in the following form

$$\left|\int_{-\infty}^{\infty} X_1(f)X_2(f)df\right|^2 \le \int_{-\infty}^{\infty}|X_1(f)|^2\,df\int_{-\infty}^{\infty}|X_2(f)|^2\,df$$

with equality holding if and only if $X_1(f) = kX_2^*(f)$.

a. Make use of Schwarz's inequality to show that

$$\left|\int_{-\infty}^{\infty} G(jf)S(f)e^{j2\pi ft}df\right|^2 \le \int_{-\infty}^{\infty}|G(jf)|^2\,df\int_{-\infty}^{\infty}|S(f)|^2\,df$$

Remember that $\left|re^{j\theta}\right| = r$

b. Make use of Schwarz's inequality and the result from part (a) to show that

$$(SNR)_o \leq \frac{2}{N_o} \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

c. From part (b) we see that the maximum possible value of $(SNR)_o$ is

$$(SNR)_o = \frac{2}{N_o} \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

Now show that

$$(SNR)_o = \frac{|s_o(T_s)|^2}{E[n_o^2(t)]} = \frac{\left| \int_{-\infty}^{\infty} G(jf) S(f) e^{j2\pi ft} df \right|^2}{\frac{N_o}{2} \int_{-\infty}^{\infty} |G(jf)|^2 df}$$

is equal to this maximum when

$$G(jf) = kS\ (f) e^{-j2\pi f T_s}$$

where $S\ (f)$ is the complex conjugate of $S(f)$

4. Making use of the result from Problem (3) we have that the impulse response h(t) of the matched filter is given by

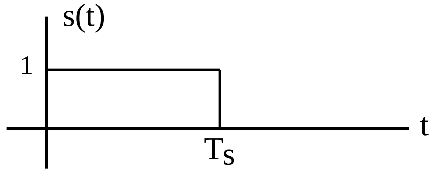
$$\begin{aligned} h(t) = F^{-1}[G(jf)] = \int_{-\infty}^{\infty} G(jf) e^{j2\pi ft} df &= \int_{-\infty}^{\infty} kS\ (f) e^{-j2\pi f T_s} e^{j2\pi ft} df \\ &= \int_{-\infty}^{\infty} kS\ (f) e^{-j2\pi f (T_s - t)} df \\ &= \left(\int_{-\infty}^{\infty} kS(f) e^{j2\pi f (T_s - t)} df \right) \\ &= s\ (T_s - t) \end{aligned}$$

as a function of s(t). Find h(t) for a matched filter if s(t) is real

5. From Problem (4) we have that the impulse response of a matched filter with a real input s(t) is given by

$$h(t) = s(T_s - t)$$

Now suppose that s(t) is a pulse as follows

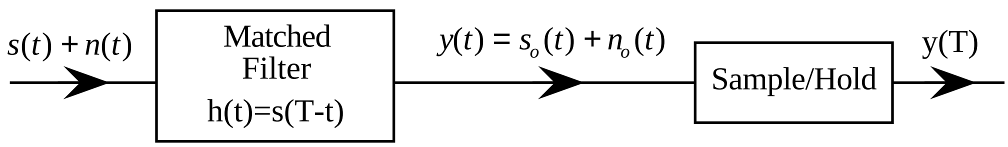


and the detector is sampling at $T_S = 10$

a. Sketch $h(t)$ of the matched filter
b. Find the frequency response of the matched filter as given by $G(jf) = F[h(t)]$
c. Sketch the magnitude of your frequency response from part (b)

d. Make use of the fact that $y(t) = h(t) \quad s(t)$ to sketch y(t)
e. From part (d) we see that y(t) at the output of the matched filter at time T_S is equal to the integral of s(t) from 0 to T_S. Explain why integration reduces the affects of noise

6. The objective of this problem is to show how the output of a matched filter is related to the energy of the transmitted pulses. Given a matched filter followed by a sample-and-hold as follows



with

$$h(t) = s(T-t) \qquad h(t-\tau) = s\big(T-(t-\tau)\big) = s(-t+T+\tau)$$

and

$$y(t) = x(t) \quad h(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t x(\tau)s(-t+T+\tau)d\tau$$

where T is the duration of the pulse

a. Show that when x(t) = s(t) then

$$y(T) = \int_0^T s^2(\tau)d\tau = E = \text{ Energy of s(t)}$$

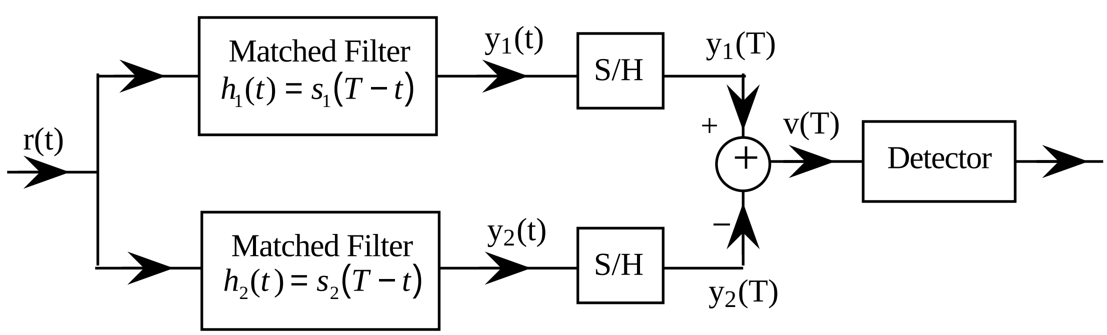
b. Show that if x(t) now contains additive noise n(t) as follows x(t) = s(t) + n(t) then

$$y(T) = E + n_o(T)$$

where

$$n_o(T) = \int_0^T n(\tau)s(\tau)d\tau$$

7. Generalizing on the result of Problem (6) we have a circuit of matched filters for detecting signals $s_1(t)$ for 1 and $s_2(t)$ for 0 as follows



a. Show that when $r(t) = s_1(t) + n(t)$ then

$$v(T) = v_1(T) = s_{01}(T) + n_o(T)$$

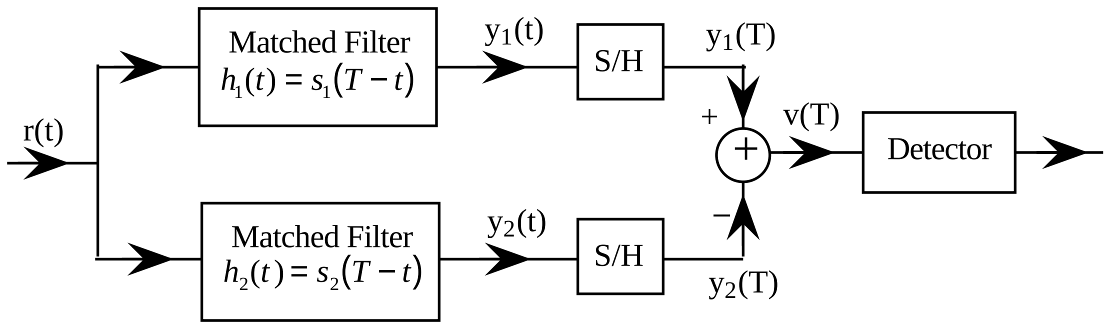
where $s_{01}(T) = \int_0^T s_1(t)\big[s_1(t) - s_2(t)\big]dt$ and $n_0(T) = \int_0^T n(t)\big[s_1(t) - s_2(t)\big]dt$

b. Show that when $r(t) = s_2(t) + n(t)$ then

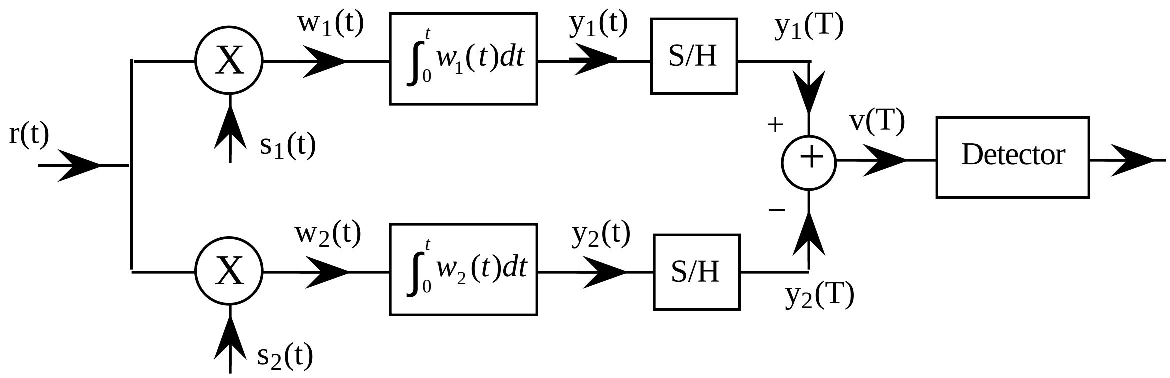
$$v(T) = v_2(T) = s_{02}(T) + n_o(T)$$

where $s_{02}(T) = \int_0^T s_2(t)[s_1(t) - s_2(t)]dt$ and $n_0(T) = \int_0^T n(t)[s_1(t) - s_2(t)]dt$

8. The objective of this problem is to show that the matched filters we developed in this Investigation as follows



are equivalent to **correlation receivers** as follows



Memorize the structure of correlation receivers. Then show that correlative receivers are equivalent to matched filter receivers - that they have the same values of $v(T)$ for the same inputs - by showing that

a. When $r(t) = s_1(t) + n(t)$ then

$$v(T) = v_1(T) = s_{01}(T) + n_o(T)$$

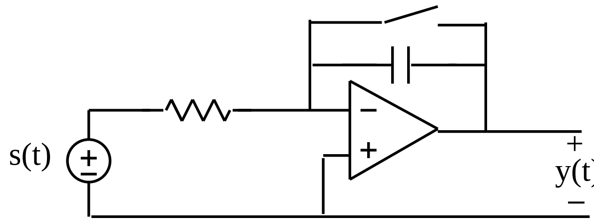
where $s_{01}(T) = \int_0^T s_1(t)[s_1(t) - s_2(t)]dt$ and $n_0(T) = \int_0^T n(t)[s_1(t) - s_2(t)]dt$

b. And when $r(t) = s_2(t) + n(t)$ then

$$v(T) = v_2(T) = s_{02}(T) + n_o(T)$$

where $s_{02}(T) = \int_0^T s_2(t)[s_1(t) - s_2(t)]dt$ and $n_0(T) = \int_0^T n(t)[s_1(t) - s_2(t)]dt$

9. Verify that the following circuit can be used as the integrator in the correlation receiver



with the switch open until time T at which time it's closed. Why do we close the switch at time T. Note that we call this an *integrate-and-dump* circuit