ECE 409 - BASEBAND TRANSMISSION - INVESTIGATION 6
INTRODUCTION TO EQUALIZERS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we showed how raised cosine filters can reduce intersymbol interference in bandlimited communication channels. This is all great but in general the receiver still needs to deal with the distortion - especially the phase distortion - caused by the frequency responses $G_C(jf)$ of nonideal communication channels as follows

The objective of this Investigation is to introduce equalizers - filters that "undo" the distortion caused by the channel.

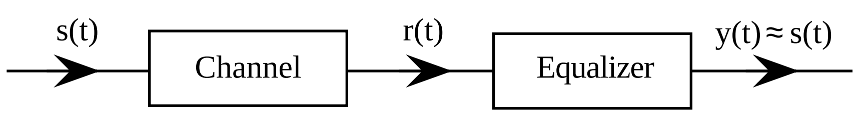
1. As we said in the introduction the nonideal frequency responses $G_C(jf)$ of channels like telephone lines can in general cause distortion in the transmission of signals. The objective of this problem is to illustrate the affects of phase distortion. Suppose in particular that the following channel

causes changes in the phases of the transmitted sinusoids s(t) as follows

$$s(t) = \cos(2\pi 100t) + \cos(2\pi 200t + 0.2)$$
$$r(t) = \cos(2\pi 100t - 0.4) + \cos(2\pi 200t + 0.7)$$

a. Use Mathcad or an equivalent to obtain a graph of the transmitted signal $s(t)$
b. Find the phases $G_C(jf)$ of the channel's frequency response at $\mathrm{f} = 100\,\mathrm{Hz}$ and $\mathrm{f} = 200\,\mathrm{Hz}$
c. Use Mathcad or an equivalent to obtain a graph of the received signal $r(t)$
d. Describe the distortion caused by the phase changes
e. What can you conclude about the affects of phase distortion

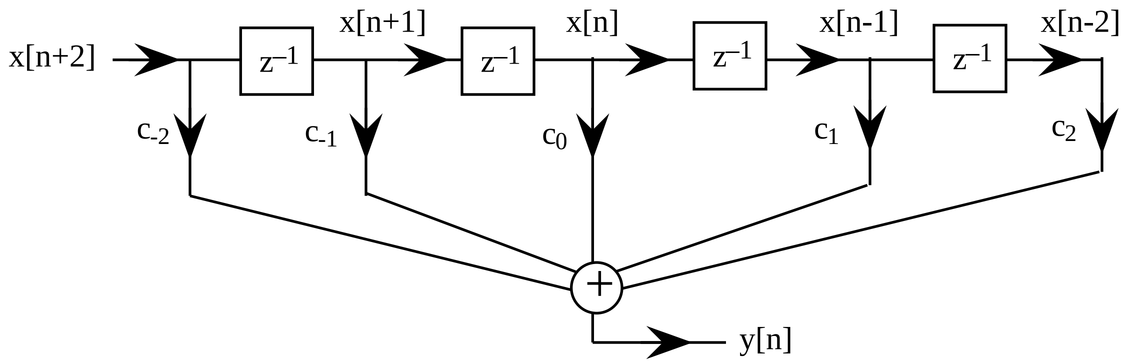
2. In Problem (1) we showed how a channel can distort a signal by changing the phases of its sinusoids. The objective of an **equalizer** in a circuit like the following



is to get us back $s(t)$ from $x(t)$ as much as possible by "undoing" the affects of the channel. Find the phase $G_E(jf)$ of the equalizer at $\mathrm{f} = 100\,\mathrm{Hz}$ and at $\mathrm{f} = 200\,\mathrm{Hz}$ when

$$s(t) = \cos(2\pi 100t) + \cos(2\pi 200t + 0.2)$$
$$x(t) = \cos(2\pi 100t - 0.4) + \cos(2\pi 200t + 0.7)$$

3. To keep the math and the implementation of equalizer filters as simple and reliable as possible they're usually implemented as FIR (Finite Impulse Response) filters like the following



Note that the c_k's are the impulse response of the FIR filter

   a. Write the equation for y[n]
   b. Find y[2] if $x[-2]=3$, $x[-1]=2$, $x[0]=0$, $x[1]=-1$, $x[2]=1$ and $c_{-2}=2$, $c_{-1}=1$, $c_0=1.5$, $c_{-1}=1$, $c_2=2$

4. FIR equalizers like those in Problem (3) are great but with $y[n]$ equal to only a finite number of terms $G_E(jf)$ is not going to be able to completely "undo" the phase and magnitude distortion caused by $G_C(jf)$. We need to settle for choosing the coefficients c_k to in some "way" minimize the difference between the transmitted signal $s(t)$ and the output of the equalizer $y(t)$. A common approach is to choose the coefficients c_k of the FIR filter to minimize the expected value of the square of the error as follows

$$\varepsilon = E\left[\left(y(t)-d(t)\right)^2\right]$$

The reason for this choice of error function is that it's relatively straightforward to find its minimum by taking its derivative and setting it to zero.

   a. Make use of the fact that

$$y(t) = \sum_{n=-N}^{N} c_n x\left(t-nT_b\right)$$

   to show that setting the derivative of ε to zero as follows

$$\frac{\partial \varepsilon}{\partial c_m} = 0 \qquad m = 0, \pm 1, \ldots, \pm N$$

   gives us $E\left[\left(y(t)-d(t)\right)x\left(t-mT_b\right)\right] = 0 \qquad m = 0, \pm 1, \ldots, \pm N$

   b. Show that your results in part (a) imply that the following autocorrelations are equal

$$R_{yx}\left(mT_b\right) = R_{dx}\left(mT_b\right) \qquad m = 0, \pm 1, \ldots, \pm N$$

   c. If we now substitute in our equation for $y(t)$ and do some rearranging we have from parts (a) and (b) that

$$\begin{bmatrix} R_{xx}(0) & R_{xx}(T_b) & \cdots & R_{xx}(2NT_b) \\ R_{xx}(T_b) & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ R_{xx}(2NT_b) & R_{xx}((2N-1)T_b) & \cdots & R_{xx}(0) \end{bmatrix} \begin{bmatrix} c_{-N} \\ \vdots \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} R_{xd}(-NT_b) \\ \vdots \\ \vdots \\ R_{xd}(NT_b) \end{bmatrix}$$

We call these the **Wiener-Hopf equations** for the coefficients c_k. Write out the Wiener-Hopf matrix equation for $N = 1$

5. The Wiener-Hopf equations of Problem (4) are great except that calculating the autocorrelation coefficients in real time can be difficult - especially in applications involving modems where new coefficients are needed everytime a new connection is made. So we need something more practical that gives good results. One alternative - the **Least Mean Squares (LMS) algorithm** developed by Widrow and Hoff - gives us a way to **adaptively** update the coefficients c_k of the FIR equalizer as new data is inputted as follows

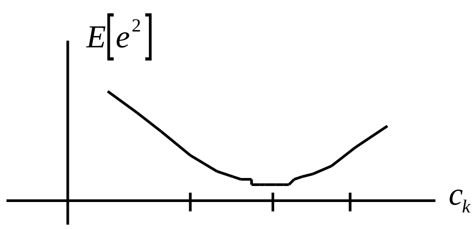
$$c_k[n+1] = c_k[n] - \frac{1}{2}\mu g_k$$

where $c_k[n]$ is the present value of the coefficient c_k, $c_k[n+1]$ is its next value and g_k is given by the following expression

$$g_k = \frac{\partial}{\partial c_k} E\left[e^2[n]\right] = \frac{\partial}{\partial c_k} E\left[\left(s[n] - y[n]\right)^2\right]$$

Note that $\mu > 0$ is a parameter that scales how much the coefficients c_k change from one sample time to the next.

a. Explain in words what information g_k gives us
b. Why is the term containing g_k subtracted from $c_k[n]$. Illustrate with graphs. Hint - if $g_k > 0$ then the error will increase as c_k increases and if $g_k < 0$ then . . . as illustrated in the following graph



6. The basic idea of the LMS algorithm is to approximate the following expectation

$$g_k = \frac{\partial}{\partial c_k} E\left[e^2[n]\right] = \frac{\partial}{\partial c_k} E\left[\left(s[n] - y[n]\right)^2\right] = 2E\left[\left(s[n] - y[n]\right)\frac{\partial}{\partial c_k} y[n]\right] = 2E\left[e[n]x[n-k]\right]$$

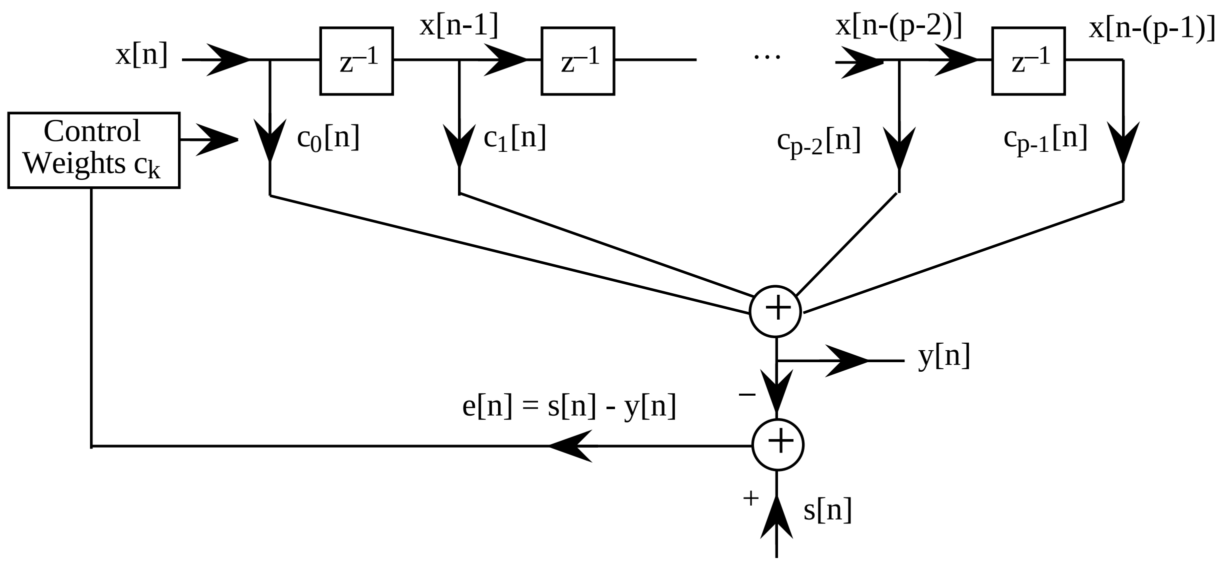
by the following

$$g_k = 2E\left[e[n]x[n-k]\right] \approx 2e[n]x[n-k]$$

We therefore have

$$c_k[n+1] = c_k[n] - \frac{1}{2}\mu g_k = c_k[n] - \mu e[n]x[n-k]$$

with $c_k[n+1]$ - the value of the coefficient c_k at the (n+1)st iteration - equal to a function of known parameters $c_k[n]$, $x[n-k]$ and $e[n] = s[n] - x[n]$ as indicated in the following adaptive equalizer



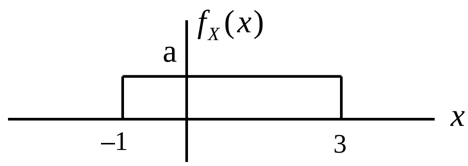
Now to actually implement the LMS algorithm the equalizer needs to know the signal $s[n]$ being transmitted in order to calculate the error $e[n]$. So before sending any real data the transmitter sends a signal $s[n]$ that the equalizer knows the value of so it can calculate $e[n]$. Note that this must be done every time a new connection is made

a. Draw an adaptive FIR equalizer with $p = 3$ at $n = 0$
b. Now make use of the LMS algorithm to find $c_k[1]$ for $k = 0,1,2$ if $p = 3$, $c_k[0] = 0$, s[0] = 1, x[0] = 0.9, x[-1] = 0.1, x[-2] = 0. Use $\mu = 0.2$. Hint - first find y[0] and then e[0] and then substitute into the equation for $c_k[1]$

7. Circuit Review - Given a linear circuit
   a. What do we mean by the impulse response $h(t)$ of the circuit
   b. How is $h(t)$ related to the transfer function $G(jf)$ of the circuit

8. Probability Review - Given a continuous random variable X and its probability density function $f_X(x)$
   a. What is $P = \int_a^b f_X(x)dx$ the probability of
   b. What is $\int_{-\infty}^{\infty} x f_X(x)dx$ equal to
9. Probability Review -
   a. What are random variables
   b. Give an example of a random variable
   c. What information does the discrete probability distribution $f_X(k)$ give us when $k = 1$

10. Probability Review - Given the following continuous uniform probability density



Find

a. a
b. $P(1 \leq x \leq 2)$
c. $P(x \geq 1)$