ECE 409 - BASEBAND TRANSMISSION - INVESTIGATION 4
POWER SPECTRAL DENSITIES - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we showed that the power spectral densities of deterministic signals $x(t)$ can be obtained by calculating the Fourier Transforms of their autocorrelations as follows

$$S_x(f) = \lim_{T \to \infty} \frac{1}{T} \left| X_T(f) \right|^2 = F\left[R_x(\tau) \right]$$

where the autocorrelation of the power signal $x(t)$ is given by

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt = \langle x(t)x(t+\tau) \rangle$$

The objective of this Investigation is to extend these results to the case of *random* baseband signals of the following form

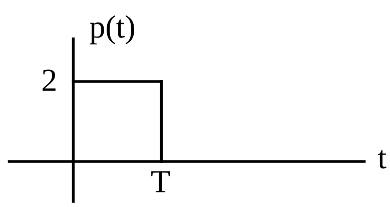
$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

where the $a_n's$ are from the *random* data being transmitted

1. Let us begin by plotting some baseband signals of the following form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

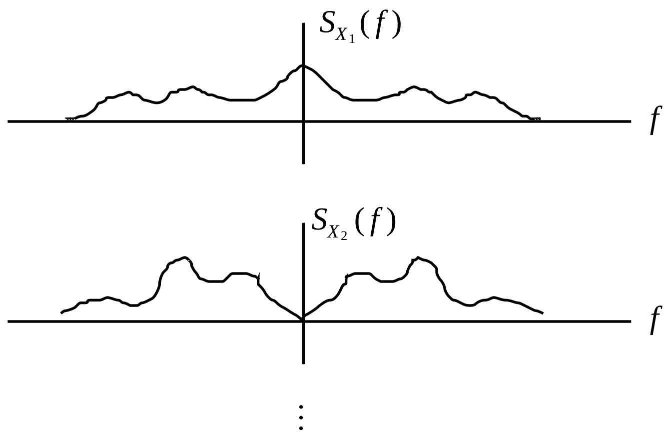
   with pulses p(t) as follows



   and coefficients a_n equal to random binary sequences of 1's and -1's from the data. Sketch $x(t)$ if

   a. $a_{-2} = 1, \quad a_{-1} = -1, \quad a_0 = -1, \quad a_1 = 1, \quad a_2 = -1$
   b. $a_{-2} = -1, \quad a_{-1} = -1, \quad a_0 = 1, \quad a_1 = -1, \quad a_2 = 1$

2. Whenever we have a set of random signals $x_1(t), x_2(t), \ldots$ like in Problem (1) with different power spectral densities like the following



we simply define the power spectral density $S_X(f)$ of the *ensemble* to be the expectation - the average of $S_{X_1}(f), S_{X_2}(f), \cdots$. In particular at each frequency f we have

$$S_X(f) = E\left[\lim_{T\to\infty} \frac{1}{T}\, |X_T(f)|^2\right] = \lim_{T\to\infty} \frac{1}{T}\, \frac{E\left[|X_T(f)|^2\right]}{T}$$

Now suppose we have have an ensemble of three signals with power spectral densities at $f = 1000\,Hz$ as follows

k	$S_{X_k}(1000)$
1	4
2	5
3	7

where the power spectral densities are in units of mW/Hz. Find the power spectral density $S_X(1000)$ at $f = 1000\,Hz$ if

a. All the signals are equally likely
b. $f_k(1) = 0.5$ and $f_k(2) = f_k(3) = 0.25$

3. In the previous Problem we calculated the power spectral density of a simple discrete ensemble at a particular frequency. The objective of this Problem is to introduce a general method for finding $S_X(f)$. Going through the math it can be shown that $S_x(f)$ can be obtained as follows

   (1) First find the autocorrelation of the ensemble of random signals as follows

   $$R_X(t+\tau, t) = E\left[x(t)x(t+\tau)\right]$$

   by finding the expected value of $x(t)x(t+\tau)$ at each time t for each value of τ

   (2) The power spectral density of the random signals can then be shown to equal

   $$S_x(f) = F\left[\langle R_X(t+\tau, t)\rangle\right]$$

   where $\langle R_X(t+\tau, t)\rangle$ = time average of $R_X(t+\tau, t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_X(t+\tau, t)\, dt$. Note that if the random process is wide sense stationary then

$$R_X(t+\tau, t) = R_X(\tau) \quad \text{and so} \quad S_x(f) = F\left[R_X(\tau)\right]$$

Suppose $x(t) = 10\cos(2\pi f_o t + \theta)$ where θ is uniformly distribution between 0 and 2π

a. Find $R_X(t+\tau, t)$
b. Find $S_X(f)$

4. What are the pros and cons of defining power spectral densities of random signals as an expectation like we did in Problem (2)

5. We now apply our method from Problem (3) for finding $S_x(f)$ to baseband signals of the following form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

where p(t) is the baseband pulse and the a_n's are a random sequence of binary numbers. Going through the math it can be shown that if $x(t)$ is *wide sense stationary* then

$$R_X(\tau) = E\left[x(t)x(t+\tau)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} R_A(m) R_P(\tau - mT)$$

where $R_A(m) = E\left[a_n a_{n+m}\right]$ = autocorrelation of the random sequence a_n

$R_p(\tau) = \int_{-\infty}^{\infty} p(t)p(t+\tau)dt$ = autocorrelation of the deterministic pulse p(t)

Be sure to note that there is no 1/T in front of the autocorrelation integral for the pulse $p(t)$ since it's an energy signal

Now suppose that a_n is a random sequence of 1's and –1's like the following

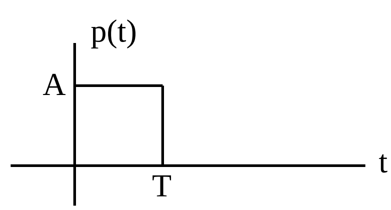
1, –1, –1, 1, –1, 1, 1, . . .

with P(1) = P(-1) = 0.5

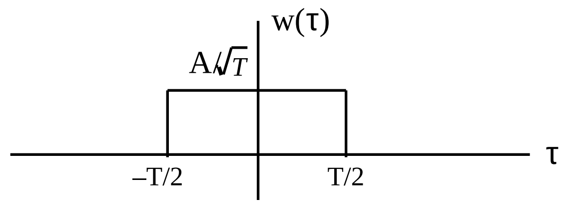
a. Show that $R_A(m) = \begin{cases} 1 & m = 0 \\ 0 & otherwise \end{cases}$

b. Make use of your result in part (a) to show that $R_X(\tau) = \frac{1}{T} R_P(\tau)$

c. Now sketch $R_X(\tau)$ when p(t) is a pulse as follows



d. Make use of your result in part (c) to show that $S_X(f) = F\left[R_X(\tau)\right] = A^2 T \operatorname{sinc}^2(Tf)$. Hint - note that $R_X(\tau)$ is equal to the convolution of the following signal with itself



e. Make use of your result in part (d) to sketch $S_X(f)$

6. In Problem (5) we calculated the power spectral densities of random pulse trains of the following form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

by first calculating their autocorrelations $R_X(\tau) = E\left[x(t)x(t+\tau)\right] = \frac{1}{T}\sum_{m=-\infty}^{\infty} R_A(m)R_P(\tau - mT)$ and then taking their Fourier Transforms to obtain $S_X(f) = F\left[R_X(\tau)\right]$

In this problem we make use of the general result that if $x(t)$ is a random pulse train of the form

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

then it can be shown that

$$S_X(f) = F\left[R_X(\tau)\right] = \frac{1}{T}|P(f)|^2 \sum_{m=-\infty}^{\infty} R_A(m) e^{-j2\pi mft}$$

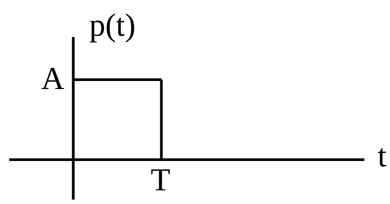
where $P(f)$ is the Fourier Transform of the pulse $p(t)$

a. Make use of this result for $S_X(f)$ to show that $S_X(f) = \frac{1}{T}|P(f)|^2$ for the pulse train of Problem (3)
b. Find and sketch P(f)
c. Make use of your results in parts (a) and (b) to find and sketch $S_X(f)$
d. Confirm that your results in this and Problem (3) are the same

7. Now suppose x(t) is a baseband signal as follows

$$x(t) = \sum_{n=-\infty}^{\infty} a_n p(t - nT)$$

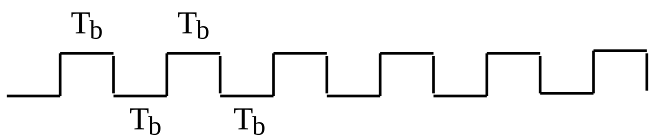
with p(t) a rectangular pulse as follows



with a_n's equal to random sequences of the values $\{-2,\ -1,\ 1,\ 2\}$ - all of which are equally probable

a. Sketch a typical signal x(t)
b. Find the autocorrelation of x(t)
c. Find the power spectral density of x(t)

8. In the previous problems we showed how to calculate the power spectral densities of random baseband signals. The objective of this problem is to find the minimum bandwidth required to transmit a baseband signal x(t) in the "worst case" when it alternates between 1 and 0 at every clock pulse as follows



Suppose in particular that x(t) is being sampled at the rate of f_s samples/sec with B bits/sample. Then data is being transmitted at the rate of

$$f_b = Bf_s \text{ bits/sec}$$

with

$$T_b = \frac{1}{f_b} \text{ secs/bit}$$

And so x(t) has a period $2T_b$ and therefore frequency

$$\frac{1}{2T_b} = 0.5 f_b$$

Now a communications channel passing such a signal must be able to pass at least the fundamental. And so the bandwidth required for the channel must satisfy

$$BW \geq 0.5 f_b \text{ Hz}$$

Which means we can, at least in principle, transmit 2W bits/sec through a channel of bandwidth W. We call 2W bits/sec the *channel capacity*.

a. Find the minimum bandwidth required to pass an analog signal sampled at the frequency f_s = 8 KHz with 8 bits per sample
b. How many bits/sec can be transmitted through a channel of bandwidth $W = 10\ KHz$