ECE 409 - BASEBAND TRANSMISSION - INVESTIGATION 3
POWER SPECTRAL DENSITY - PART I

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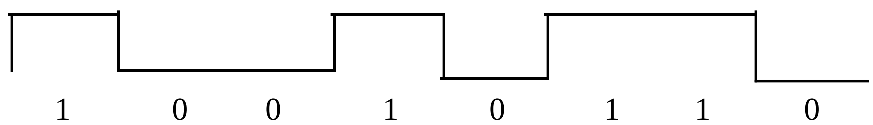
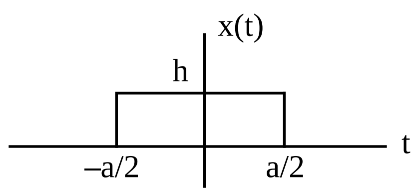
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this and the next six Investigations is to analyze the transmission of 1's and 0's by *baseband signals* consisting of pulses like the following



where a high voltage represents a 1 and a low voltage a 0. The objective of this and the next Investigation in particular is to find the *power spectral densities* of these signals for random sequences of 1's and 0's like those found in communication systems. We begin in this Investigation with deterministic baseband signals.

1. As we all know Fourier Transforms are great for signals like the following pulse



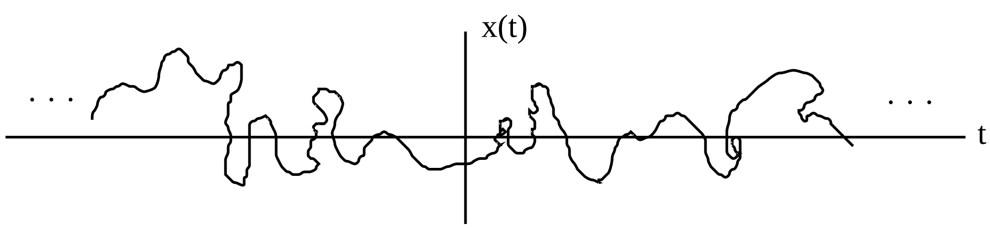
and for periodic signals like pulse trains

   a. Find and sketch the Fourier Transform of the above pulse
   b. Find the "bandwidth" of the pulse in part (a) equal the frequency to the first *null* - the first frequency where $X(f) = 0$
   c. Find and sketch the Fourier Transform of $y(t) = x(t)\cos(2\pi bt)$
   d. Generalize on your result in part (b) to find the bandwidth of $y(t)$ in part (c)

2. As we saw in Problem (1) the Fourier Transform works great for signals like pulses and modulated pulses. But it can be shown that the Fourier Transform as follows

$$\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt$$

does not in general exist - the limit does not converge - for nonperiodic signals like the following



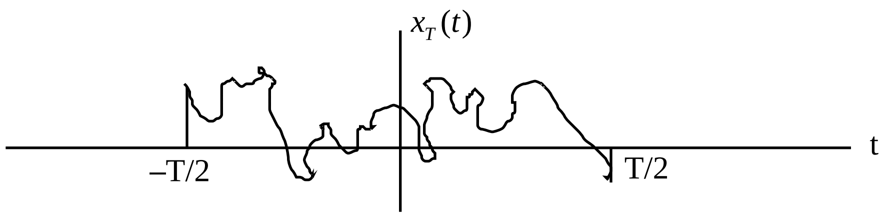
that are not *square integrable* as follows

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

a Show that $x(t) = e^t \quad t \geq 0$ is not square integrable and therefore does not have a Fourier Transform

b. Find your own example of a signal that is not square integrable

3. One way to find the "bandwidths" of signals like those in Problem (2) that are not square integrable is to take the Fourier Transforms of "representative" truncated samples $x_T(t)$ like the following



But this takes special care and fine tuning. Alternatively we can calculate the *power spectral densities* of deterministic signals $x(t)$ as long as $x(t)$ is a **power signal** - has finite average power as follows

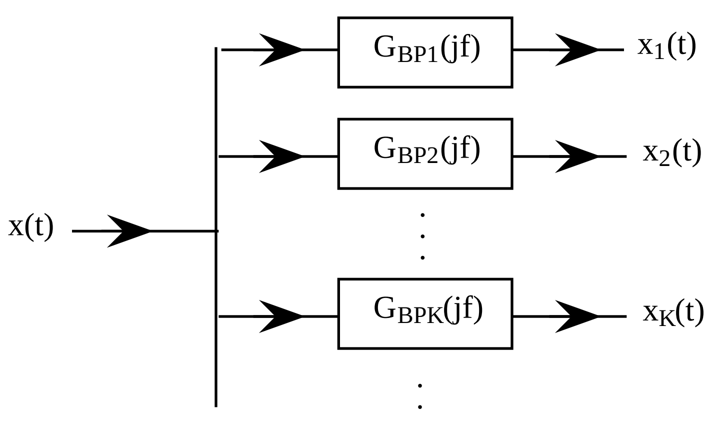
$$0 < P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)dt = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt < \infty$$

a. Explain why P_{av} is the average *normalized power* - the average rate at which energy is dissipated in a 1Ω resistor with voltage $x(t)$

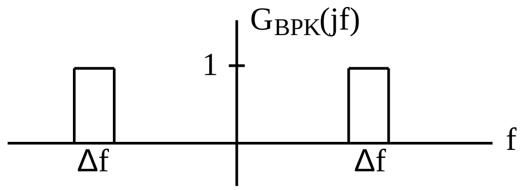
b. Show that $x(t) = A\cos(2\pi bt)$ is a power signal. In particular find the corresponding P_{av}

4. The objective of this and the next problem is to define **power spectral density** $S_x(f)$ introduced in Problem (3). If $x(t)$ is a power signal then we can obtain its power spectral density as follows

   (1) Put $x(t)$ through a bank of bandpass filters as follows



with bandwidths equal to Δf as follows



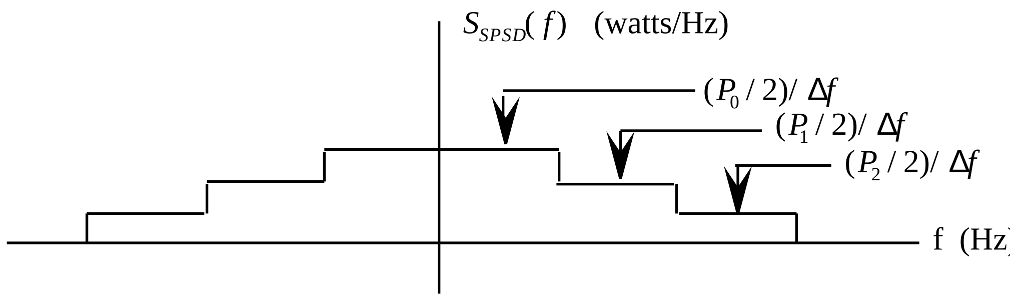
(2) Find the average power P_k at the output of each of the filters as follows

$$P_k = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_k^2(t)dt$$

(3) Find the power spectral density in each interval as follows

$$S_{PSD}(k) = \frac{P_k}{\Delta f} = \frac{\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_k^2(t)dt}{\Delta f}$$

(4) Combine the power spectral densities to form a *double-sided staircase power spectral density* $S_{SPSD}(f)$ as follows

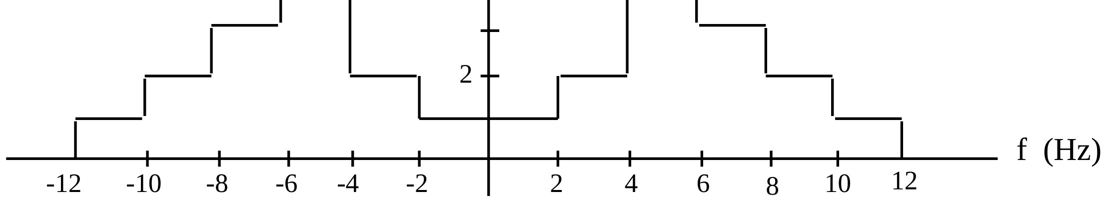


a. Find and sketch the staircase power spectral density $S_{SPSD}(f)$ of a signal $x(t)$ with P_0 = 5 watts, P_1 = 3 watts, P_2 = 2 watts and Δf = 2 Hz

b. Now by superposition the total average power is the sum of all the average powers of $x_1(t)$, $x_2(t)$, . . . and so is equal to the integral of the staircase power spectral density as follows

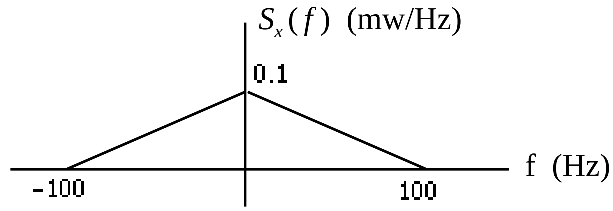
$$P_{av} = \int_{-\infty}^{\infty} S_{SPSD}(f)df$$

Make use of this result to find the total average power P_{av} of a signal with the following power spectral density



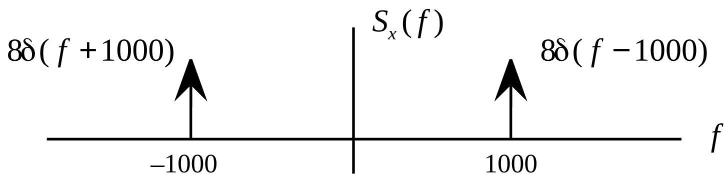
5. As we make the passbands Δf of the bandpass filter narrower and narrower the staircase power

spectral density plots metamorphize into nice smooth power spectral densities $S_x(f)$ like the following



Find the total average power P_{av} of the signal $x(t)$ with this power spectral density.

6. Explain why the power spectral density of the sinusoid $x(t) = 4\cos(2\pi 1000t)$ is as follows



7. Power spectral densities are particularly nice not only because they are straightforward to measure but also because we can obtain closed form expressions for them for baseband and passband digital communications signals. To calculate these power spectral densities we begin with the fact that average power P_{av} is equal to the following limit

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)dt$$

where $x_T(t)$ is a truncated $x(t)$. But $x_T(t)$ is an **energy signal** - a signal that delivers a finite amount of energy as follows

$$E = \int_{-\infty}^{\infty} p(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

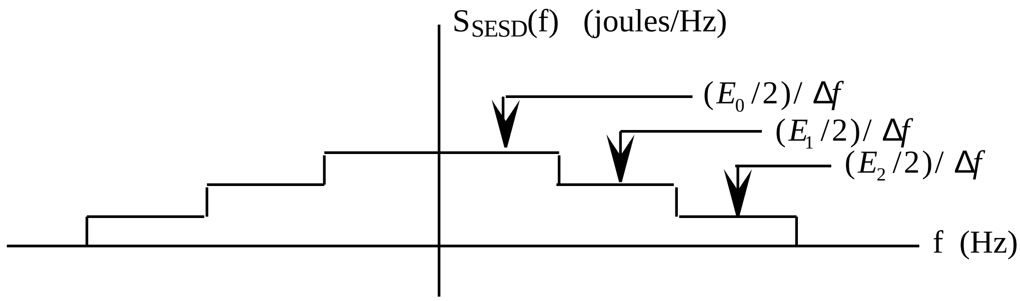
to a 1Ω resistor. The objective of this and the next several problems is to develop what we mean by *energy spectral density* and then use the results to find an expression for calculating the power spectral density

   a. Explain why energy signals are not power signals
   b. Verify that the following pulse is an energy signal

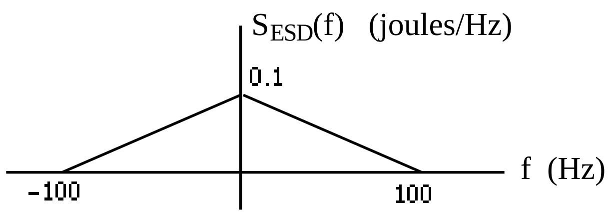
   c. Verify that the unit step function u(t) is not an energy signal
   d. Come up with your own example of an energy signal

8. The **energy spectral density** of an energy signal is basically the same as the power spectral density of a power signal except that it gives us the energy density as a function of frequency rather than the average power as a function of frequency. We begin, as we did for the power

spectral density, by passing the signal through a bank of bandpass filters to obtain energies E_k which we then make use of to obtain two-sided staircase energy spectral densities as follows



Now taking the limit as $\Delta f \to 0$ just like we did in the case of power signals we obtain energy spectral density plots like the following

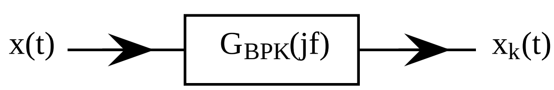


Find the total energy of the signal x(t) with this energy spectral density.

9. The objective of this Problem is to make use of *Parseval's Theorem* for energy signals $x(t)$ as follows

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

to obtain an expression for the energy spectral density of $x(t)$ in terms of its Fourier Transform $X(f)$. From Parseval's Theorem we see that the energy of $x_k(t)$ at the output of the k'th bandpass filter as follows



is given by

$$E_k = \int_{k\Delta f}^{(k+1)\Delta f} |X(f)|^2 df$$

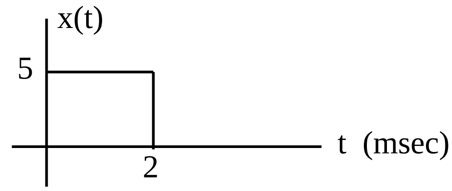
and so the energy spectral density in this interval is given by the following expression

$$\frac{E_k}{\Delta f} = \frac{\int_{k\Delta f}^{(k+1)\Delta f} |X(f)|^2 df}{\Delta f} \approx \frac{|X(f_k)|^2 \Delta f}{\Delta f} = |X(f_k)|^2$$

when Δf is small. So in the limit as $\Delta f \to 0$ we see that the energy spectral density of an energy signal $x(t)$ is equal to the square of the magnitude of its Fourier Transform as follows

$$E_{ESD}(f) = |X(f)|^2$$

Make use of this result to find and sketch the energy spectral density of the following pulse



10. The objective of this problem is to find an expression for the energy spectral density as the Fourier Transform of an autocorrelation. Taking the inverse Fourier Transform of

$$E_{ESD}(f) = |X(f)|^2$$

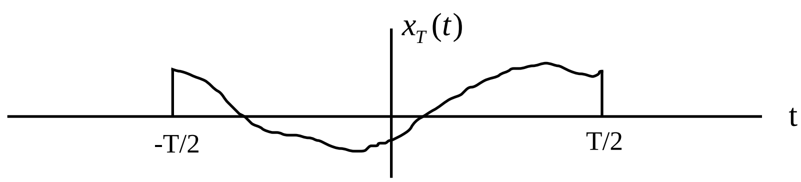
we obtain the result

$$F^{-1}\left[E_{ESD}(f)\right] = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt = R_X(\tau)$$

where $R_X(\tau)$ is the **autocorrelation** of the **energy signal** $x(t)$. Therefore

$$E_{ESD}(f) = F\left[\int_{-\infty}^{\infty} x(t)x(t+\tau)dt\right] = F\left[R_X(\tau)\right]$$

11. The objective of this and the next problem is to make use of our results on energy spectral densities to find the power spectral density $S_X(f)$ of a power signal $x(t)$. The trick is to begin by truncating the power signal $x(t)$ to obtain the following energy signal



for which we know by *Parseval Theorem* that

$$\int_{-T/2}^{T/2} x_T^{\,2}(t)dt = \int_{-\infty}^{\infty} |X_T(f)|^2 df$$

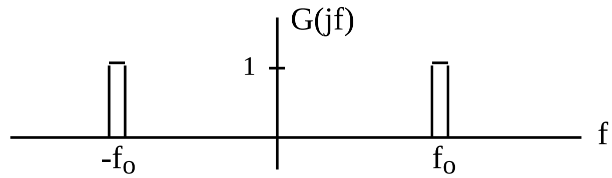
where $X_T(f)$ is the Fourier Transform of the truncated $x(t)$ as follows. Now since the average power P_{av} of $x(t)$ is given by

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_T^{\,2}(t)dt$$

we have that

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 df$$

If we now pass x(t) through a bandpass filter as follows



of center frequency f_o and narrow bandwidth Δf then the signal at the output of the filter will have an average power of value

$$\lim_{T\to\infty}\frac{1}{T}\int_{f_o-\Delta f/2}^{f_o+\Delta f/2}\left|X_T(f)\right|^2 df \approx \Delta f \lim_{T\to\infty}\frac{1}{T}\left|X_T(f_o)\right|^2$$

Make use of this result to show that the power spectral density $S_X(f)$ of $x(t)$ is given by

$$S_X(f) = \lim_{T\to\infty}\frac{1}{T}\left|X_T(f)\right|^2$$

12. In the previous problem we showed how to find the power spectral density $S_X(f)$ of a **power signal** $x(t)$ in terms of its Fourier Transform as follows

$$S_X(f) = \lim_{T\to\infty}\frac{1}{T}\left|X_T(f)\right|^2$$

The objective of this problem is to make use of this relationship to find $S_X(f)$ in terms of its **autocorrelation** $R_x(\tau)$ as follows

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

Be sure to note that the autocorrelation of a power signal differs from that of an energy signal in that it is multiplied by a factor of $1/T$.

Now the trick to finding $S_X(f)$ in terms of $R_x(\tau)$ is to simply take the inverse Fourier Transform of $S_X(f) = \lim_{T\to\infty}\frac{1}{T}\left|X_T(f)\right|^2$ to obtain

$$F^{-1}\left[S_X(f)\right] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt = R_x(\tau)$$

and so

$$S_x(f) = F\left[R_x(\tau)\right]$$

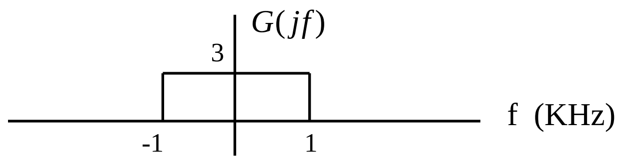
We call this the **Wiener-Khinchine** theorem. **Memorize** it. Then apply it to $x(t) = \cos(2\pi bt)$ as follows

a. Find $R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_T \cos(2\pi bt)\cos(2\pi b(t+\tau))dt$
b. Make use of your result in part (a) to find and plot $S_x(f)$
c. Make use of your result in part (b) to find the average power P_{av} of $x(t)$
d. Verify that your result in part (c) is in fact the average power of $x(t)$

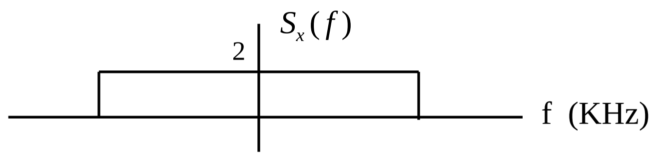
13. The objective of this problem is to see what happens to the power spectral density of a signal

when it goes through a linear circuit. Given a linear system as follows

with the following transfer function



a. Make use of the result $S_y(f) = |G(jf)|^2 S_x(f)$ to sketch the power spectral density of y(t)



b. Make use of the your result in part (a) to find the average power of y(t)