ECE 409 - ERROR CORRECTING CODES - INVESTIGATION 20
INFORMATION THEORY AND CHANNEL CAPACITY

SUMMER 2004 **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last several Investigations we introduced block and convolution coding for the detection and correction of errors. The objective of this Investigation is to introduce the ideas of how much information a signal contains and channel capacity - the amount of information a given channel can pass.

1. From the sampling theorem we know that if we sample a bandlimited signal fast enough we can - at least in principle - reconstruct the signal from its samples. But if we first transmit the signal we'll find that the reconstructed signal will be a distorted version of the original as a result of the samples being corrupted by noise.

   So converting our signals to binary before transmission - which amounts to quantizing them - does not inherently introduce more distortion than would otherwise occur. As we've said before the basic advantage of binary signals is that it takes a lot of noise to cause an error. But we must still guard against errors because an error in a significant bit has the potential of causing problems.

   The objective of this problem is to define what we mean by the **information** of a digital signal as introduced by Hartley and Nyquist at Bell Labs in the 1920's. There are two basic ideas underlying the definition as follows

   (1) If the probability of a message m is equal to one then the amount of information of the message is zero - we already knew what the result was going to be. If on the other hand the probability of a message m is 0 then it contains an infinite amount of information - we'd be very surprised to get it

   (2) If we receive two independent messages then the total information we receive should be the sum of their informations.

   From these two constraints comes the following definition for the information of a signal m

$$I_m = \log_2\left(\frac{1}{p(m)}\right)$$

   a. Why is the information a function of 1/p
   b. Why is the log function used. Hint - make use of the definition to find the total information of two independent message signals m_1 and m_2 with probabilities $P(m_1)$ and $P(m_2)$
   c. Why is the log to the base 2
   d. Find the information of a 5 bit message if every message is equally likely
   e. Find the information of an n bit message if every message is equally likely

2. The **entropy H** of a set of independent messages is the average value of their information as follows

$$H = \sum p(m)\log_2\left(\frac{1}{p(m)}\right) \text{ bits/message}$$

a. Verify that the entropy H of a 1 bit message is given by

$$H = p\log_2\left(\frac{1}{p}\right) + (1-p)\log_2\left(\frac{1}{1-p}\right) \text{ bits/message}$$

b. Sketch the entropy H in part (a) as a function of p. Describe and explain what's going on at p=0 and p=1
c. Where is H maximum

3. What's the entropy H of n bit messages that are equally likely

4. The objective of this problem is to see how the presence of noise affects the amount of information that can be transmitted through channels by signals that can in general take on more than just two values
   a. How many bits of information are contained in signals that can take on M different values
   b. Shannon and Hartley showed a receiver can detect at most

$$M = \frac{S+N}{N} = 1 + \frac{S}{N}$$

distinct signals without error where S is the average power of the source and N the average power of the noise. How much information I is contained in any such signal assuming all signals are equally likely

5. The objective of this problem is to introduce **channel capacity** - the ability of a channel with bandwidth B and average noise power N to transmit information. Assuming the signals in Problem (4) are each transmitted for T seconds and thus require bandwidths on the order of B as follows

$$B = \frac{1}{T}$$

then the channel capacity is

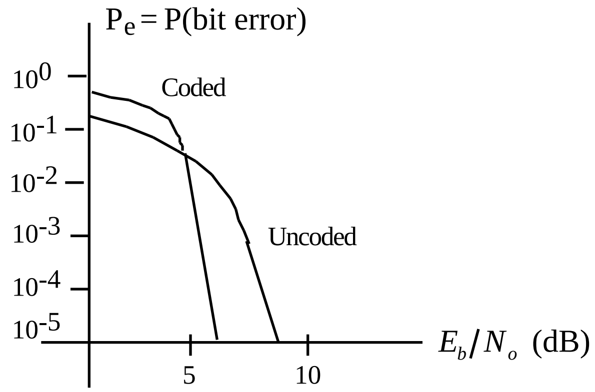
$$C = \frac{I}{T} = B\log_2\left(1 + \frac{S}{N}\right)$$

We call C the **Shannon-Hartley** capacity limit for error free communication. It's the most information in bits/sec that can be transmitted with no errors through a communication channel of bandwidth B with an ideal frequency response and additive white Gaussian noise (AWGN). As a result C is an upper bound with which to compare our designs. Describe how C is related to
   a. S = average power of the signal
   b. N = average power of the noise
   c. Bandwidth B. Note that $N = BN_o$

6. Find the channel capacity C for a telephone line with SNR=30dB and B=3.5KHz. Note that the term S/N in the equation for C is the ratio S/N - it's not S/N in dB

7. The goal of the Investigations on coding has been to add extra bits to reduce the errors and as a

result get closer to the Shannon-Hartley limit. The results of analysis and/or simulation typically look as follows



where E_b/N is the average power of each bit over the average power per Hz of the noise.

a. Describe the curves
b. Why do coded signals have higher probabilities of error when E_b/N is small