ECE 409 - SIGNAL SPACE ANALYSIS - INVESTIGATION 17
INTRODUCTION TO OPTIMUM RECEIVERS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From the last Investigation we know how to write general bandpass modulated signals as sums of basis functions as follows

$$s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)$$

and how to use correlators to retrieve the coefficients s_{jk}. The objective of this Investigation is to introduce how to optimally detect $s_i(t)$ - how to maximize the probability that the detector is correct in the presence of additive white Gaussian noise.

1. We begin with a problem to review signal spaces. Suppose $s_i(t)$ is a sum of N basis functions as follows

   $$s_i(t) = s_{i1}\phi_1(t) + s_{i2}\phi_2(t) + \cdots + s_{iN}\phi_N(t)$$

   with $\mathbf{s_i} = (s_{i1}, s_{i2}, \ldots, s_{iN})$ equal to the signal vector. And M is the number of signals $s_i(t)$. Come up with a possible set of signal vectors if

   a. $N = 1$ and $M = 2$
   b. $N = 2$ and $M = 4$

2. The objective of this problem is to review conditional probability. Come up with a conditional probability problem where $P(A \mid B) < P(A)$

3. The objective of this problem is to see how conditional probabilities can be used in the design of detectors. Suppose a communication system is trying to make use of the fact that the signal $r(t)$ was received to decide whether the signal actually sent was $m_1(t)$ or $m_2(t)$. Which signal would you say was sent based on the following conditional probabilities. Justify your answer

   $$P\big(m_1(t) \text{ sent} \big| r(t) \text{ received}\big) = 0.4 \qquad P\big(m_2(t) \text{ sent} \big| r(t) \text{ received}\big) = 0.6$$

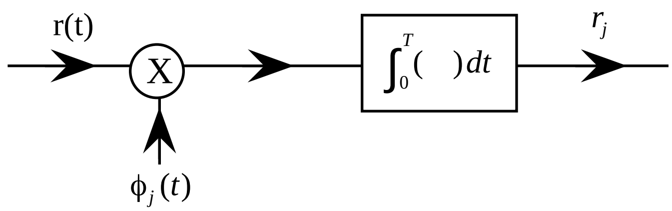
4. We now begin the process of showing how Bayes' Rule can be used to design optimum receivers. What is Bayes' Rule all about. Illustrate with an example

4. What is meant by *a priori* and *a posteriori* probabilities. Illustrate with an example

5. Going through the analysis we can show that if the received signal is distorted by additive Gaussian white noise with power spectral density $N_o/2$ as follows

   $$r(t) = s_i(t) + n(t)$$

   then the output of a receiver's correlator as follows



will equal $r_j = s_{ij} + n_j$ where $n_j = \int_0^T n(t)\phi_j(t)dt$ is Gaussian with probability density as follows

$$f_{n_j}(\alpha_j) = \frac{1}{\sqrt{\pi N_o}} \exp\left(-\frac{\alpha_j^2}{N_o}\right)$$

So as usual the received signal $r(t)$ won't in general equal the signal $s_i(t)$ that was transmitted. How would you make use of the conditional probabilities

$$P(m_k \mid r)$$

to in general make a "best guess" of which signal was actually sent

6. If our detector chooses the m_k that maximizes $P(m_k \mid r)$ - the one for which

$$P(m_k \mid r) > P(m_i \mid r) \qquad \text{for all } i \neq k$$

as its best estimate of the signal that was transmitted then we call the receiver a **maximum a posteriori receiver (MAP receiver)** or **optimum receiver**. **Memorize** these terms. Suppose in particular that our MAP receiver chooses $\hat{m} = m_k$ when the received signal is $r = \rho$

a. Explain why

$$P(\text{Correct Decision } \mid r = \rho) = P(m_k \mid r = \rho)$$

is the probability that the MAP receiver makes the correct decision

b. Explain why

$$P(\text{Correct Decision }) = \int_{-\infty}^{\infty} P(\text{Correct Decision } \mid r = \rho) f_r(\rho) d\rho$$

is the probability that a MAP receiver makes the correct decision

7. Starting from Bayes Rule it can be shown that optimum receivers find $\hat{m}$ by finding the value of s_i that maximizes

$$N_o \ln\left[P(m_i)\right] - \left|r - s_i\right|^2$$

where the following vectors are represented in terms of their basis functions

$\mathbf{r} = (r_1, r_2, \ldots, r_N) =$ output of correlator
$\mathbf{s_i} = (s_{i1}, s_{i2}, \ldots, s_{iN}) =$ signal vector of i'th signal
$\left|r - s_i\right|^2 = (r - s_i)\ (r - s_i) =$ dot product of $(r - s_i)$ with itself
$=$ square of the distance between r and s_i

Describe in words what $\hat{m}$ depends on

8. Given the results of Problem (7)
   a. Make use of the fact that $\left|r - s_i\right|^2 = (r - s_i)\ \ (r - s_i)$ to show that finding the s_i that maximizes $N_o \ln\left[P(m_i)\right] - \left|r - s_i\right|^2$ is equivalent to finding the s_i that maximizes

$$\mathbf{r}\ \ \mathbf{s_i} + c_i$$

   where

$$\mathbf{r}\ \ \mathbf{s_i} = \sum_{j=1}^{N} r_j s_{ij} = \text{ dot product of } \mathbf{r} \text{ and } \mathbf{s}$$

$$c_i = \frac{N_o}{2}\ln\left[P(m_i)\right] - \frac{1}{2}\,|\mathbf{s_i}|^2 \qquad \text{where } \ |\mathbf{s_i}|^2 = \mathbf{s_i}\ \ \mathbf{s_i}$$

   b. Draw a block diagram of an optimum receiver for a signal space with $N = 2$ and $M = 4$ consisting of $N = 2$ correlators, a block for doing the $M = 4$ dot products and a block for finding the maximum $\mathbf{r}\ \ \mathbf{s_i} + c_i$

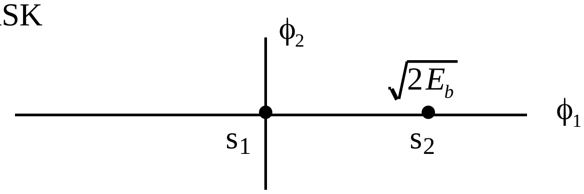
9. If all the probabilities are equal then our optimum receiver reduces to what we call a **maximum likelihood (ML) receiver**.
   a. What are the probabilities $\mathrm{P(m_i)}$ when there are M equally likely signals
   b. Why does the optimization problem for ML receivers reduce from finding the s_i that maximizes $N_o \ln\left[P(m_i)\right] - \left|r - s_i\right|^2$ to finding the s_i that minimizes

$$\left|r - s_i\right|^2$$

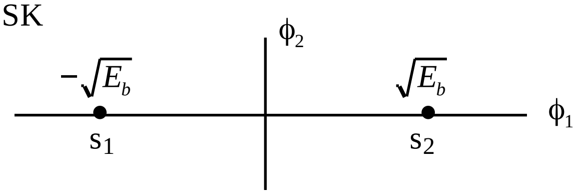
   c. Make use of your result in part (b) to explain why the optimum detector finds the s_i that is closest to r

10. Given a bandpass modulation system with $M = 2^2 = 4$ and $N = 2$ with four signals $s_1 = (1,1),\ \ s_2 = (-1,1),\ \ s_3 = (2,3),\ \ s_4 = (2,-1)$
   a. Sketch the signal space
   b. Find the signal most likely to have been sent if all the signals are equally likely and the received signal is $r = (1,-1)$

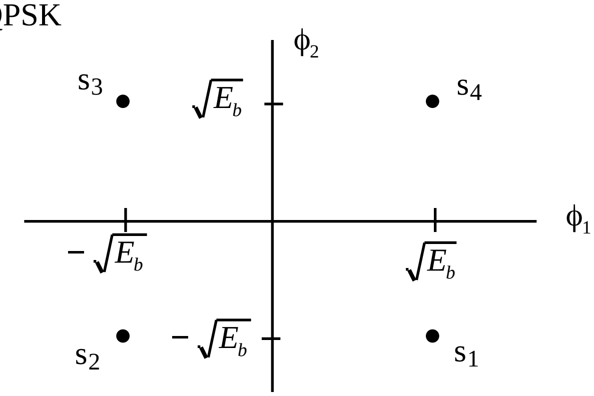
11. From Problem (10) we know that ML receivers choose the s_i that's closest to the output r of the correlator. Make use of this result to identify and describe the regions for each s_i in the following constellations. Draw the boundaries
   a. BASK



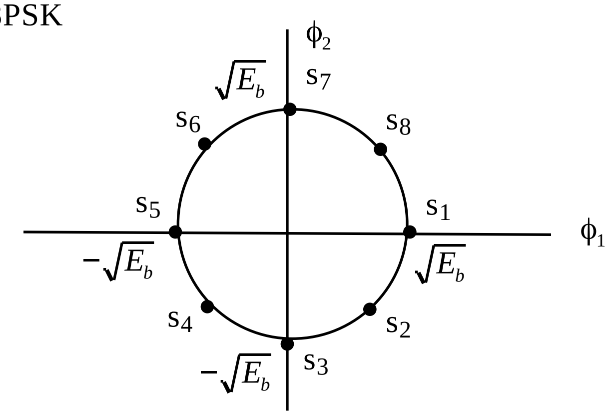
b. BPSK



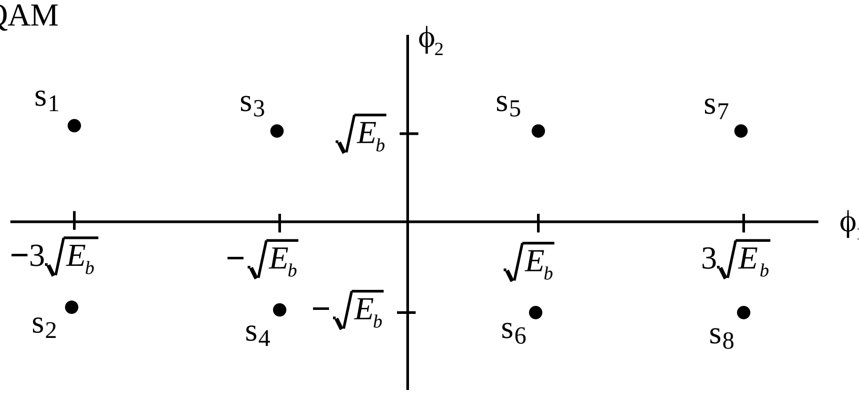
c. QPSK



d. 8PSK



e. 8 QAM



12. Show that if two equally likely points in a constellation are a distance *d* apart then the probability P_e that one will be mistaken for the other in the presence of zero mean white Gaussian noise with variance $\sigma_o^2 = N_o/2$ is

$$P_e = Q\left(\frac{d}{\sqrt{2N_o}}\right)$$

13. Make use of the result for P_e from Problem (12) to find P_e for each of the following binary signals. Then verify that your results agree with our previous results
    a. BASK
    b. BPSK
    c. BFSK

14. Make use of the result for P_e from Problem (12) to show that P_e for QPSK is given by

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_o}}\right) - \left[Q\left(\sqrt{\frac{2E_b}{N_o}}\right)\right]^2$$

Hint - first find the probability of no error for a given s_i. Assume that the affects of the noise on the coefficients s_{ij} is independent