ECE 405 - ANALOG COMMUNICATIONS - INVESTIGATION 8
INTRODUCTION TO FREQUENCY MODULATION - PART I

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of the last two Investigations was to introduce amplitude modulation and see how message signals m(t) can modulate the amplitudes of carrier sinusoids. Our main goals were to see what AM signals look like in the time domain and to find their spectrums so we could find their bandwidths. The objective of this and the next Investigation is to introduce angle modulated signals as follows

$$s(t) = A_c \cos\left(2\pi f_c t + \theta_a(t)\right)$$

in which $m(t)$ modulates the phase and frequency of the carrier.

1. The objective of this first problem is to introduce a special case of **angle modulation** which we refer to as **phase modulation (PM)** in which the phase $\theta_a(t)$ of the carrier signal as follows

   $$s(t) = A_c \cos\left(2\pi f_c t + \theta_a(t)\right)$$

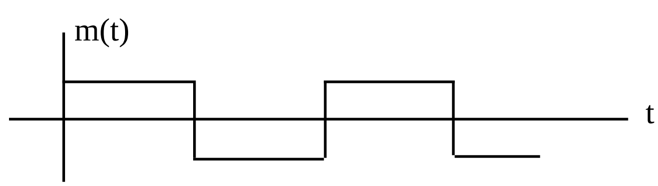
   is proportional to the message signal m(t) as follows

   $$\theta_a(t) = k_p m(t)$$

   The phase modulated signal is then

   $$s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p m(t)\right)$$

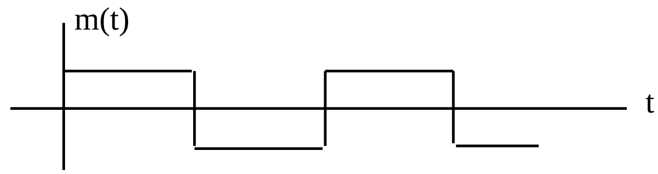
   where k_p = phase sensitivity in rad/volts. Sketch $s(t)$ if $m(t)$ is given by



2. The objective of this problem is to introduce another special case of angle modulation called **frequency modulation (FM)** in which the instantaneous frequency $f_i(t)$ of the modulated sinusoid $s(t) = A_c \cos\left(2\pi f_c t + \theta_a(t)\right)$ varies varies as a function of the message signal $m(t)$ as follows

   $$f_i(t) = f_c + k_f m(t)$$

   where k_f is the frequency sensitivity. Sketch s(t) for m(t) as follows



3. What are two ways in which the graph of an AM signal is different from that of an FM signal

4. From Problem (2) we know that for an FM signal the instantaneous frequency is given by

$$f_i(t) = f_c + k_f m(t)$$

Therefore its instantaneous phase $\theta_i(t)$ is given by

$$\theta_i(t) = 2\pi \int f_i(t)dt = 2\pi f_c t + 2\pi k_f \int m(t)dt$$

and so

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \theta_a(t)\right) = A_c \cos\left(\theta_i(t)\right) = A_c \cos\left(2\pi f_c t + 2\pi k_f \int m(t)dt\right)$$

Therefore for frequency modulation, $\theta_a(t)$ is proportional to the integral of the message signal $m(t)$ as follows

$$\theta_a(t) = 2\pi k_f \int m(t)dt \qquad \Leftrightarrow \qquad \frac{d\theta_a(t)}{dt} = 2\pi k_f m(t)$$

in contrast to phase modulation with

$$\theta_a(t) = k_p m(t)$$

Given $m(t) = A_m \cos\left(2\pi f_m t\right)$ find the equation for the

a. PM signal
b. FM signal

5. Find the average normalized power of the following general angle modulated signal

$$s(t) = A_c \cos\left(2\pi \mathrm{f_c t} + \theta_a(t)\right)$$

6. The objective of this and the next two problems is to learn about the bandwidths of FM signals by examining the special case of a sinusoidal message signal $m(t) = A_m \cos\left(2\pi f_m t\right)$ with

$$f_i(t) = f_c + k_f A_m \cos\left(2\pi f_m t\right) = f_c + \Delta f \cos\left(2\pi f_m t\right)$$

where $\Delta \mathbf{f} = \mathbf{k_f A_m}$ = **frequency deviation** of the FM signal

a. What are the minimum and maximum values of $f_i(t)$
b. Why do we call Δf the frequency deviation of the FM signal
c. Find $\theta_i(t) = 2\pi \int_0^t f_i(\tau)d\tau$
d. From part (c) we have

$$s(t) = A_c \cos\left(\theta_i(t)\right) = A_c \cos\left(2\pi f_c t + \beta \sin\left(2\pi f_m t\right)\right)$$

Why do we call $\beta = \dfrac{\Delta f}{f_m}$ the **phase deviation** of the FM signal

e. Find β for an FM signal with $k_f = 1000$ and $m(t) = 5\cos(2\pi 2000t)$

7. The objective of this problem is to find the bandwidth of the FM signal from Problem (6) with $m(t) = A_m \cos(2\pi f_m t)$ as follows

$$s(t) = A_c \cos(\theta_i(t)) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right)$$

in the special case where the phase modulation β is small compared to one radian

a. Make use of the following trig identity

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

to express s(t) as a sum of products of sinusoids

b. Make use of the result in part (a) to show that s(t) can be approximated as follows when β is small

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t)$$

c. What is the bandwidth of s(t). Hint - make use of the following trig identity

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x+y) - \frac{1}{2}\cos(x-y)$$

d. How does the bandwidth of a narrowband FM signal compare with that of an AM signal with $m(t)$ equal to a sinusoid

8. The objective of this problem is to find the bandwidths of wideband FM signals - FM signals with β much larger than one for the special case of $m(t) = A_m \cos(2\pi f_m t)$. The trick is to use Euler's Relation to express $s(t)$ as the real part of a complex exponential as follows

$$\begin{aligned} s(t) &= A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_m t)\right) \\ &= \mathrm{Re}\left[A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_m t))}\right] \\ &= \mathrm{Re}\left[A_c e^{j\beta \sin(2\pi f_m t)} e^{j2\pi f_c t}\right] \\ s(t) &= \mathrm{Re}\left[A_c x(t) e^{j2\pi f_c t}\right] \end{aligned}$$

where $x(t) = e^{j\beta \sin(2\pi f_m t)}$

a. Show that x(t) is periodic with period $T = 1/f_m$. In particular show that $x(t+T) = x(t)$

b. Since x(t) is periodic with fundamental frequency f_m we can express it in a Fourier Series expansion with coefficients X_k as follows

$$x(t) = e^{j\beta \sin(2\pi f_m t)} = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_m t}$$

with

$$X_k = \frac{1}{T}\int_{T/2}^{-T/2} x(t)e^{-j2\pi k f_m t}dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\beta\sin(2\pi f_m t)}e^{-j2\pi k f_m t}dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{-j\left(2\pi k f_m t - \beta\sin(2\pi f_m t)\right)}dt$$

If we now make the change of variable $x = 2\pi f_m t$ then we have

$$X_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j(kx - \beta\sin(x))}dx = J_k(\beta)$$

which is referred to as the Bessel function of the first kind of order k and argument β. Bessel functions do not have closed form solutions but they can easily be calculated and put in tables. Pulling all this together we see that angle modulated signals can be written as follows

$$s(t) = \mathrm{Re}\left[A_c e^{j\beta\sin(2\pi f_m t)}e^{j2\pi f_c t}\right] = \mathrm{Re}\left[A_c\left(\sum_{k=-\infty}^{\infty} A_k\, e^{j2\pi k f_m t}\right)e^{j2\pi f_c t}\right]$$

$$= \mathrm{Re}\left[A_c\left(\sum_{k=-\infty}^{\infty} J_k(\beta)\, e^{j2\pi k f_m t}\right)e^{j2\pi f_c t}\right]$$

$$= \mathrm{Re}\left[A_c\sum_{k=-\infty}^{\infty} J_k(\beta)\, e^{j2\pi(f_c + k f_m)t}\right]$$

$$= A_c\sum_{k=-\infty}^{\infty} J_k(\beta)\cos\left(2\pi(f_c + k f_m)t\right)$$

Write out the sum for $s(t)$ as a sum of sinusoids from $k = -2$ to $k = 2$. Note that $J_{-k}(\beta) = (-1)^k J_k(\beta)$

9. Given a frequency modulated signal with $A_c = 5$, $\beta = 3$, $f_m = 1000$ and $f_c = 10^4$
   a. Make use of Matlab, a calculator or a table of Bessel functions to find the coefficients $A_c J_k(\beta)$ for $k = -5$ to $k = 5$
   b. Sketch the corresponding double-sided spectral plot
   c. Write out the corresponding sum of cosines

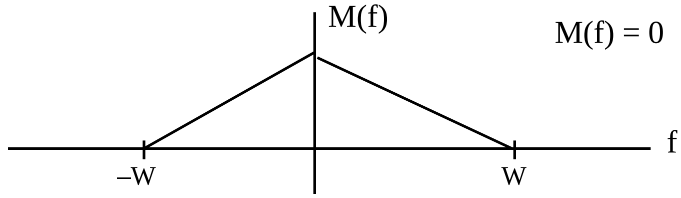
10. What does the result in Problem (9) tell us about the bandwidths of angle modulated signals in contrast to the bandwidths of AM signals.

11. From Problem (10) we know that FM signals - at least theoretically - have infinite bandwidths with spectral components that require Bessel functions to calculate. But as we would expect most of the power of an FM modulated s(t) is in the first harmonics. Luckily Carson's Formula as follows

$$B_T = 2\Delta f + 2f_m = 2\Delta f\left(1 + \frac{1}{\beta}\right) = 2f_m(\beta + 1)$$

gives us a reasonable approximation for the bandwidth of an FM signal with a sinusoidal message signal. Make use of this equation to find the bandwidth of an FM signal with carrier frequency $k_f = 1000$ and message signal $m(t) = 5\cos(2\pi 5000 t)$

12. Generalizing on the result of Problem (11) it can be shown that if the message signal $m(t)$ is **bandlimited** by W as follows



then the corresponding FM signal has a bandwidth B_T as follows

$$B_T = 2\Delta f\left(1 + \frac{1}{D}\right) \quad \text{with} \quad D = \frac{\Delta f}{W}$$

Make use of this result to find the bandwidth of an FM signal with $\Delta f = 75$ KHz as it is for commercial FM and $W = 15$ KHz. Note that commercial FM stations are allocated 200 KHz of the spectrum.