ECE 405 - REVIEW OF THE BASICS - INVESTIGATION 5
REVIEW OF THE FOURIER TRANSFORM - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we reviewed Fourier Transforms of nonperiodic signals like single pulses and showed how they can be used in frequency domain analysis. In particular we showed how to find the spectrums of signals pulses and how these spectrums can be used in analysis. Our main results were that the Fourier Transform and inverse Fourier Transform of $x(t)$ and $X(f)$ are given by

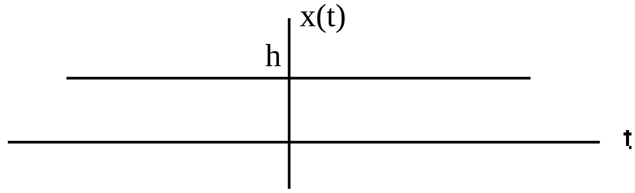
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \qquad x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

And that if $x(t)$ is the input to a linear system with transfer function $G(jf)$ then the Fourier Transform of the output $y(t)$ is related to the Fourier Transform of $x(t)$ as follows

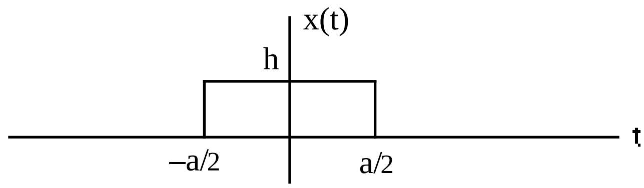
$$Y(f) = G(jf)X(f)$$

The objective of this Investigation is to review the Fourier Transforms of periodic signals and the products of signals.

1. Let us begin by finding the Fourier Transform of a constant as follows



The trick to finding the Fourier Transform of a constant is to first find the Fourier Transform of a pulse as follows



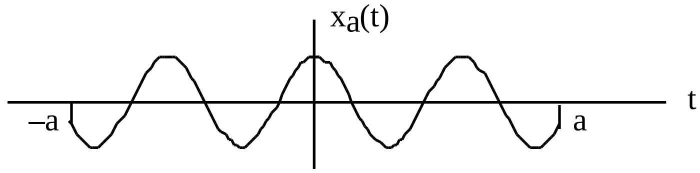
and then take the limit as the pulse width a goes to infinity. Now from the last Investigation we know that the Fourier Transform $X(f)$ of such a pulse is given by

$$X(f) = ha\,\mathrm{sinc}(fa)$$

a. Draw a sequence of graphs to show what happens to $X(f)$ as the pulse width a increases
b. Describe what's happening to the heights and widths of your graphs $X(f)$ as a increases and the pulse approaches the constant $x(t) = h$. In particular verify that your graphs in part (a) are getting closer and closer to

$$X(f) = F[h] = h\delta(f)$$

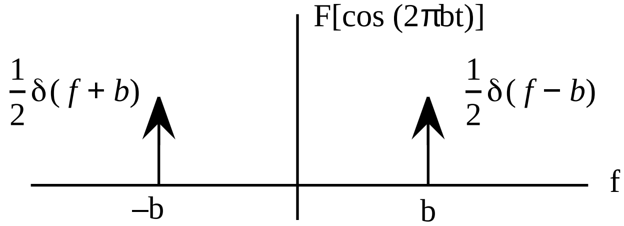
2. The objective of this and the next problem is to find the Fourier Transform of a sinusoid $x(t) = \cos(2\pi bt)$. Now the trick to finding the Fourier Transform of a sinusoid is basically the same as for finding the Fourier Transform of a constant. We start by finding the Fourier Transform of a truncated cosine $x_a(t)$ as follows



and then take the limit as *a* goes to infinity to obtain

$$F\left[\cos(2\pi bt)\right] = F\left[\frac{1}{2}e^{-j2\pi bt} + \frac{1}{2}e^{j2\pi bt}\right] = \frac{\delta(f+b)}{2} + \frac{\delta(f-b)}{2}$$

Graphically we have



Memorize this result.

a. Explain in words why the Fourier Transform of $x(t) = \cos(2\pi bt)$ is zero at every frequency except $f = b$
b. Find and sketch the Fourier Transform of $x(t) = 3\cos(2\pi 1000t)$
c. Find and sketch the Fourier Transform of $x(t) = 3\cos(2\pi 1000t) + 2\cos(2\pi 2000t)$
d. Verify that our result for the Fourier Transform of a sinusoid is the same as that for a constant when $b = 0$

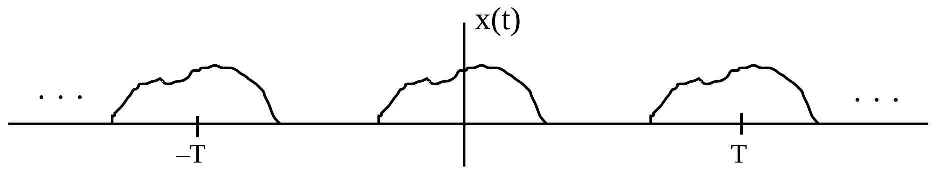
3. Generalizing on the results of Problem (2) we can make use of the fact that the Fourier Transform of a complex exponential is an impulse function as follows

$$F\left[ae^{-j2\pi bt}\right] = a\delta(f+b)$$

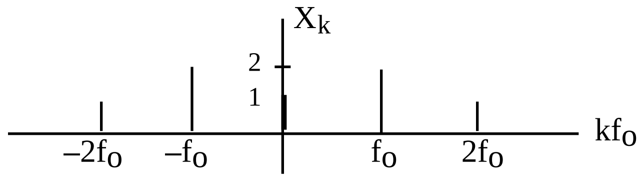
to obtain the Fourier Transform of a periodic signal $x(t)$ by taking the Fourier Transform of its Fourier Series Expansion as follows

$$F[x(t)] = F\left[\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kf_o t}\right] = \sum_{k=-\infty}^{\infty} X_k F\left[e^{j2\pi kf_o t}\right] = \sum_{k=-\infty}^{\infty} X_k \delta(f - kf_o)$$

a. Make use of these results to find the Fourier Transform $X(f)$ of a periodic signal x(t) of frequency $f_o = 1000$ Hz with Fourier Coefficients $X_0 = 2$, $X_1 = 3e^{j1.2}$, $X_2 = 2e^{-j}$
b. Sketch the magnitude of the Fourier Transform of $x(t)$ in part (a)
c. Find and sketch the Fourier Transform of the following periodic signal $x(t)$



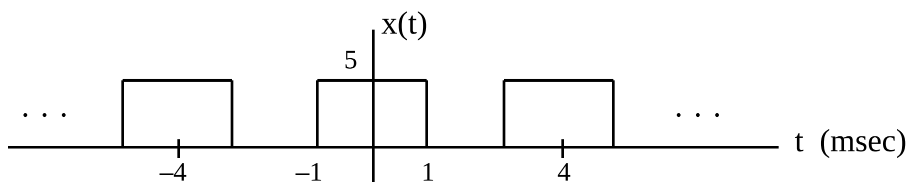
with spectral plot as follows



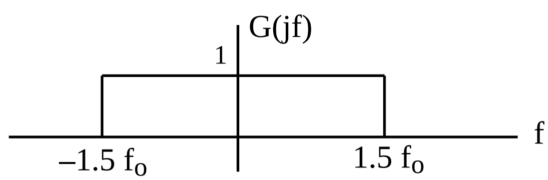
4. From the result of Problem (3) we see that we can obtain the Fourier Transform of a periodic signal x(t) as follows

   (1) Find the spectral coefficients X_k
   (2) Make use of the X_k's to obtain the Fourier Transform

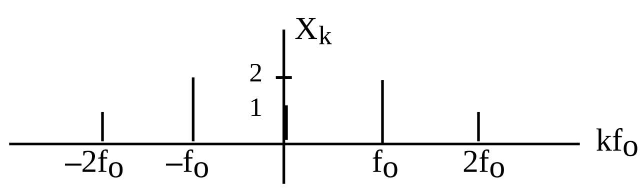
   Make use of this procedure to sketch the Fourier Transform of the following pulse train



5. Find the Fourier Transform at the output of a circuit with frequency response as follows



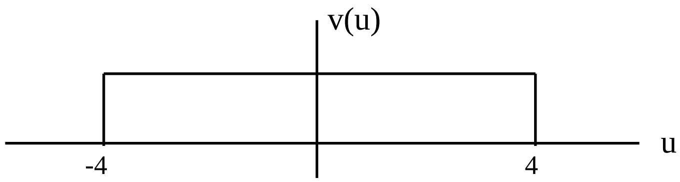
   when the input has the following spectral plot



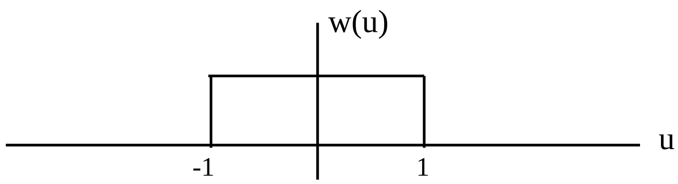
6. The objective of this and the next several problems is to review convolution as follows

$$v(u) \quad w(u) = \int_{-\infty}^{\infty} v(u) w(r - u) du$$

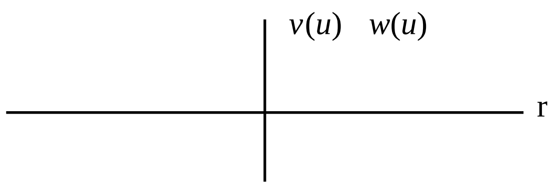
   so we can make use of it to calculate the Fourier Transforms of products. Sketch the convolution of the following two signals



and



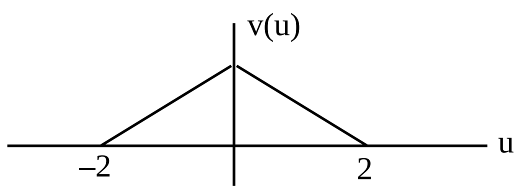
as a function of r as follows



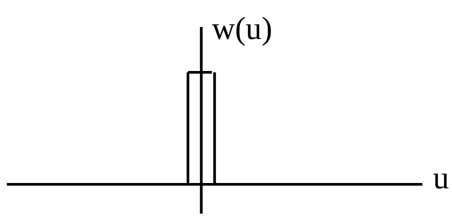
Hint - Make use of the fact that the value of the convolution at each point $r = r_o$ can be obtained as follows

(1) Plot w(-u) by flipping w(u) about the vertical axis
(2) Obtain $w(r_o\text{-}u)$ by shifting the origin of w(-u) to r_o
(3) Multiply v(u) times $w(r_o\text{-}u)$
(4) Find the area under the curve $v(u)w(r_o\text{-}u)$

7. Given the following signal v(u)



a. Sketch the convolution of $v(u)$ and $w(u)$ equal to the following narrow pulse of area one

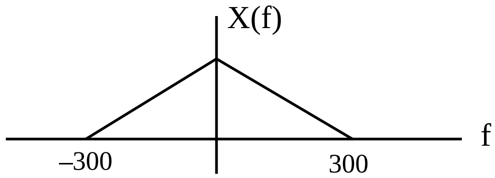


b. Describe your result in part (a)
c. Now make use of your result in part (a) to sketch the convolution of $v(u)$ and $w(u) = \delta(u)$
d. Sketch the convolution of $v(u)$ and the shifted impulse $\delta(u - 2)$
e. Describe your result in part (d)

8. **Memorize** the following result relating convolution and Fourier Transforms

$$F[x(t)y(t)] = X(f) \quad Y(f)$$

And then make use of it to sketch the following Fourier Transforms if $x(t)$ has the Fourier Transform



a. Sketch the Fourier Transform $F[x(t)\cos(2\pi 1000t)]$
b. Describe how your result in part (a) is related to $X(f)$
c. Sketch the Fourier Transform $F[x(t)(\cos(2\pi 1000t) + \cos(2\pi 2000t))]$
d. Describe how your result in part (c) is related to $X(f)$
e. Sketch the magnitude of the Fourier Transform $F[x(t)p(t)]$ where $p(t)$ is a pulse train of fundamental frequency $f_o = 1$ KHz

9. This problem makes use of convolution to find the spectrum of a product of sinusoids.
   a. Make use of convolution to find and sketch $F[\cos(2\pi 1000t)\cos(2\pi 3000t)]$. Hint - approximate the impulses by very narrow pulses.
   b. Verify that your result is consistent with the fact that

$$\cos x \cos y = 0.5 \cos(x+y) + 0.5 \cos(x-y)$$

10. In this and the last three Investigations we reviewed frequency response and Fourier
    a. List the results you understand
    b. List the results you don't understand
    c. What questions do you have