ECE 405 - REVIEW OF THE BASICS - INVESTIGATION 4
REVIEW OF THE FOURIER TRANSFORM - PART I

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From the last two Investigations we know how to find the spectrums of periodic signals and we know how to make use of this spectral information to calculate steady state responses of linear circuits to periodic inputs.

In particular we reviewed the fact that periodic signals $x(t)$ of period T and fundamental frequency f_o as follows

$$f_o = \frac{1}{T}$$

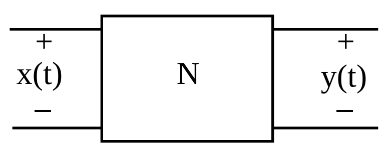
can be expressed as sums of sinusoids as follows

$$x(t) = c_o + \sum_{k=1}^{\infty} c_k \cos\left(k 2\pi f_o t + \theta_k\right)$$

where the frequencies of the harmonics are equal to integer multiples of the fundamental frequency f_o. We then reviewed how these sums of sinusoids can be expressed as sums of complex exponentials as follows

$$x(t) = c_o + \sum_{k=1}^{\infty} c_k \cos\left(2\pi k f_o t + \theta_k\right) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_o t}$$

Expressing periodic signals as sums of sinusoids or equivalently as sums of complex exponentials is particularly useful because it enables us to calculate steady state responses of linear circuits and systems to periodic inputs by simply adding up the responses to the individual harmonics. In particular the steady state response of the following linear system



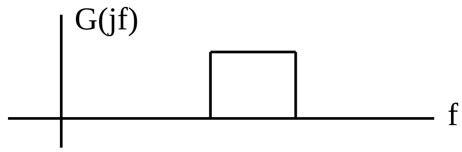
with transfer function

$$G(\mathrm{j}f) = \frac{Y(\mathrm{j}f)}{X}$$

is equal to

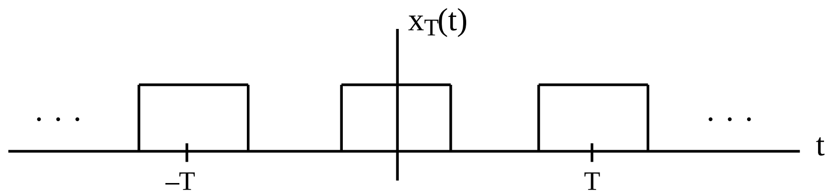
$$y(t) = \sum_{k=-\infty}^{\infty} G\left(j 2\pi k f_o\right) X_k e^{j 2\pi k f_o t}$$

We refer to this method of analyzing linear systems as **frequency domain analysis**. Frequency domain analysis is particularly useful in cases when the circuits have limited passbands like the following



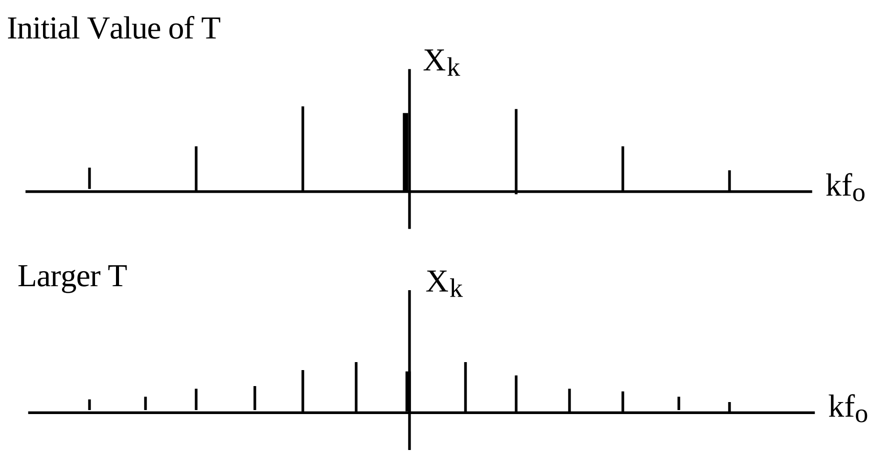
The objective of this Investigation is to review the Fourier Transform and in particular see how it can be used to find the spectrums of nonperiodic signals like single pulses and then how these spectrums can be used in frequency domain analysis analogously to what we did with periodic signals in the last Investigation.

1. We start with a periodic pulse train $x_T(t)$ as follows



like the periodic signals in the last Investigation. How does increasing the period T affect the amplitudes and spacing of the harmonics. Illustrate with drawings of spectral plots.

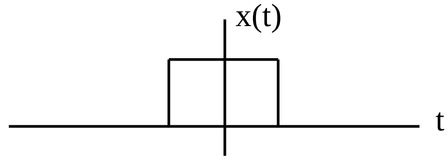
2. From Problem (1) we know that as we increase the period T of a periodic signal like a pulse train $x_T(t)$ the harmonics get closer together as they ominously get smaller as follows



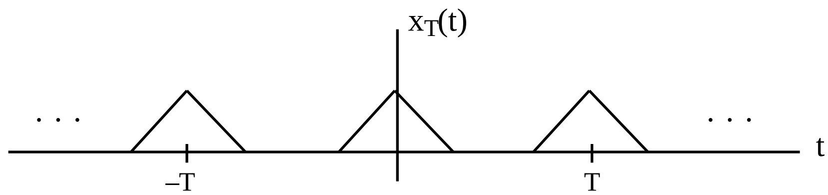
Now this does not prevent us from obtaining x(t) from its harmonics X_k as follows

$$x_T(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_o t}$$

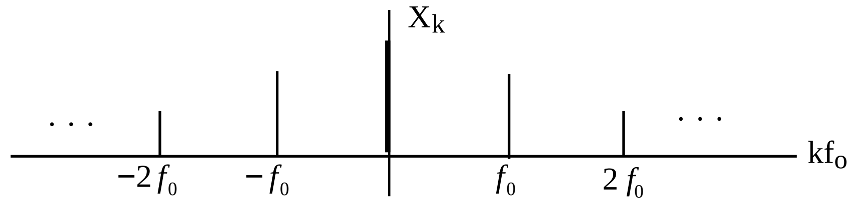
for any finite value of T. But it does prevent us from taking the limit as T goes to infinity as the pulse train *metamorphizes* into a single pulse as follows



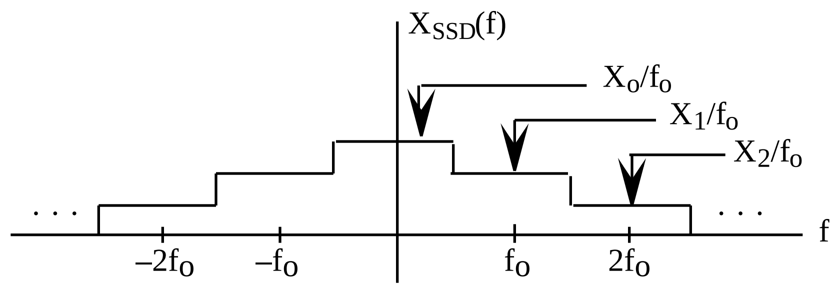
The objective of this problem is to introduce the mathematical trick of *staircase spectral* *density functions $X_{SSD}(f)$* that enable us to take this limit. Given a periodic signal of period T like the following



with spectral plot as follows



the *staircase spectral density function $X_{SSD}(f)$* is the following "continuous" signal



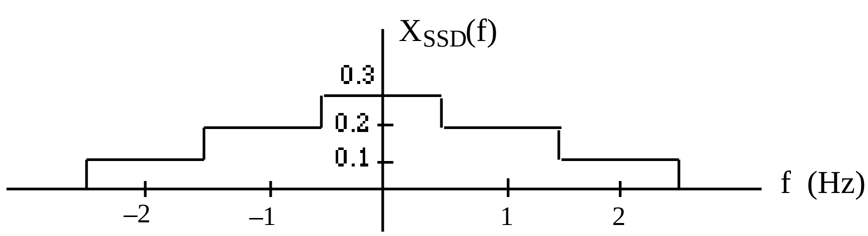
with constant values in each interval $kf_o - \frac{f_o}{2} \le f \le kf_o + \frac{f_o}{2}$ equal to

$$X_{SSD}\left(kf_o\right) = \frac{X_k}{f_o} = TX_k = T\frac{1}{T}\int_T x(t)e^{-jk2\pi f_o t}dt = \int_T x(t)e^{-jk2\pi f_o t}dt$$

a. How are the values of $X_{SSD}\left(kf_o\right)$ related to the values of the harmonics X_k

b. Find and draw the magnitude of $X_{SSD}(f)$ for the periodic signal x(t) with $X_0 = 0.04$, $X_1 = 0.02e^{j1.3}$, $X_2 = 0.01$, $X_3 = 0.02e^{j}$ and $T = 100$

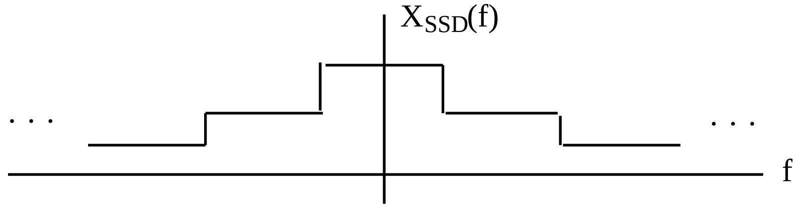
c. Find and draw the spectral plot of the periodic signal $x(t)$ with the following staircase spectral density function. What is the period T of $x(t)$



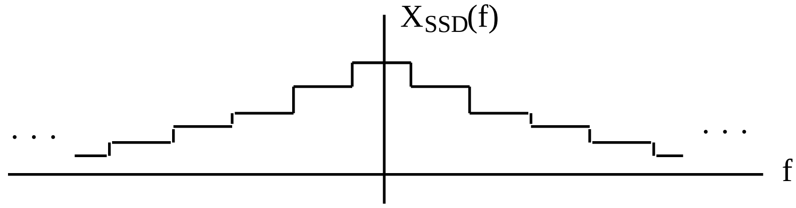
d. Show that we can express $x_T(t)$ in terms of $X_{SSD}\left(kf_o\right)$ as follows

$$x_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{SSD}\left(kf_o\right)e^{j2\pi kf_o t}$$

3. The objective of this and the next two problems is to see what happens as we increase the period T. Suppose a periodic signal $x(t)$ of period T has a staircase spectral density as follows

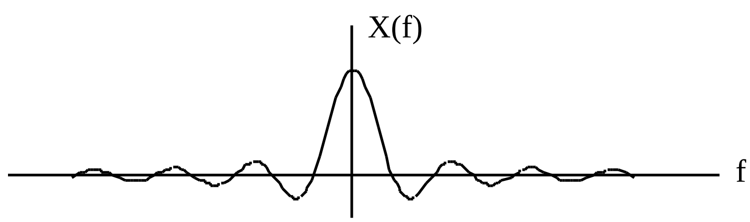


Then explain why doubling T will cause $X_{SSD}(f)$ to change as follows



but without the amplitude getting smaller.

4. If we continue increasing the period T of a periodic signal like we did in Problem (3) then the stairs of the staircase spectral density $X_{SSD}(f)$ are going to get narrower and narrower until in the limit as $T \to \infty$ the staircase spectral density becomes a nice smooth curve like the following

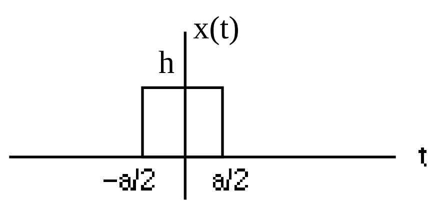


with

$$X_{SSD}(f) \quad \to \quad X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

We call X(f) the **Fourier Transform or spectral density** of x(t).

a. Make use of the integral equation for X(f) to verify that the Fourier Transform of a pulse as follows



is equal to $X(f) = ha\,\mathrm{sinc}\left(fa\right)$

b. Sketch $X(f) = ha\,\text{sinc}(fa)$ from part (a)
c. Make use of your result in parts (a) and (b) to describe what happens to the Fourier Transform (spectral density) $X(f)$ of a pulse when its width a is cut in half. What, in particular, happens to the bandwidth of the signal. Draw graphs to illustrate. Hint - find where $\text{sinc}(fa) = 0$ as a function of a.

5. From the last Investigation we know that we can obtain a periodic signal $x(t)$ from its Fourier coefficients X_k from the equation

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_o t}$$

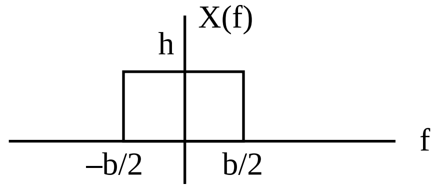
To obtain the corresponding result for the Fourier Transform we take the limit as $T \to \infty$ of

$$x_T(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_{SSD}(2\pi k f_o) e^{j2\pi k f_o t}$$

to obtain

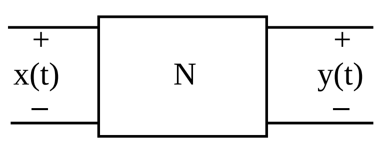
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

We call $x(t)$ the **inverse Fourier Transform** of $X(f)$.
   a. How is the equation for the inverse Fourier Transform different from that for the Fourier Transform
   b. Find $x(t)$ with the following Fourier Transform. Express your result in terms of the sinc function



   c. Sketch your result for $x(t)$ in part (b)

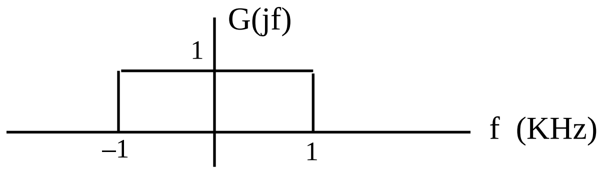
6. Now that we have Fourier Transforms for nonperiodic signals like single pulses we can use them to do frequency domain analysis of linear systems in much the same way we use Fourier Series to analyze linear systems with periodic inputs. In particular if $x(t)$ with Fourier Transform $X(f)$ is the input of the following linear system



with transfer function $G(jf)$ then the Fourier Transform of the output $y(t)$ is given by

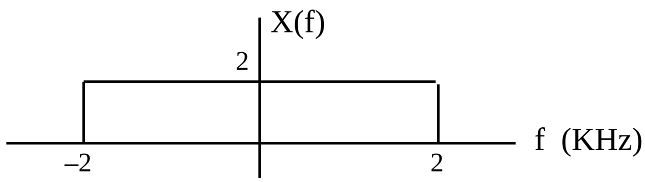
$$Y(f) = G(jf)X(f)$$

Once we have $Y(f)$ we can then use the inverse Fourier Transform to obtain $y(t)$. Make use of these results to sketch $Y(f)$ for a linear system with transfer function as follows

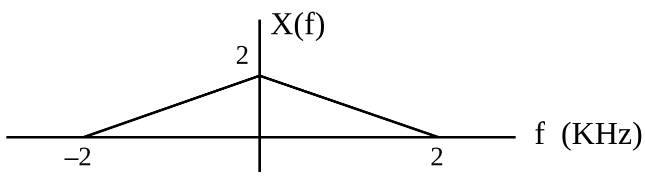


when

a. x(t) has the spectrum

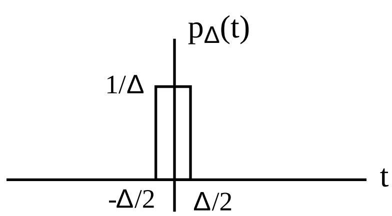


b. x(t) has the spectrum



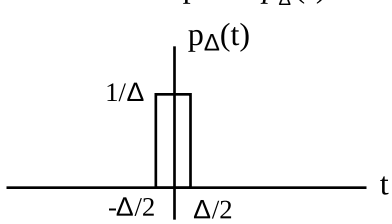
c. x(t) is a pulse of width a = 1 msec

7. The objective of this and the next several problems is to review impulse functions δ(t) so we can take their Fourier Transforms. The objective of this problem is to first sketch and then find the integrals of some unit pulses $p_\Delta(t)$ of area one with width Δ and magnitude 1/Δ as follows

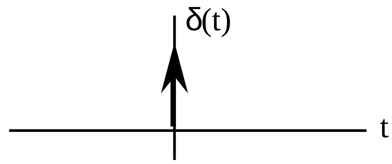


a. Sketch $p_\Delta(t-2)$
b. Sketch $p_\Delta(t+2)$
c. Find $\int_{-\infty}^{\infty} p_\Delta(t-2)dt$

8. If we now take the limit as $\Delta \to 0$ of a unit pulse $p_\Delta(t)$ as follows



we obtain what we refer to as a unit impulse function δ(t) as follows



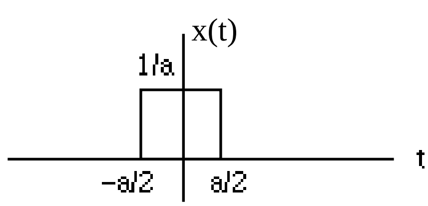
Impulse functions are important because they simplify a number of important math expressions. Now impulse functions are of course not "real functions" - they have areas equal to one despite having zero widths - but they're easy to work with because all we have to do is "think of them" as very narrow unit pulses $p_\Delta(t)$. Given this

a. Sketch $\delta(t - 2)$
b. Sketch $\delta(t + 2)$
c. Find $\int_{-\infty}^{\infty} \delta(t-2)dt$

9. The objective of this problem is to calculate the integral of an impulse times the sinusoid $x(t) = 2\cos(2\pi 1000t)$
   a. First sketch $x(t)$
   b. Sketch $p_\Delta(t)$
   c. Then sketch $x(t)p_\Delta(t)$ assuming Δ is really, really small. What is the approximate amplitude of $x(t)p_\Delta(t)$
   d. Make use of your result in part (c) to approximate $\int_{-\infty}^{\infty} x(t)p_\Delta(t)dt$
   e. Make use of your result in part (d) to find $\int_{-\infty}^{\infty} x(t)\delta(t)dt$
   f. Sketch $x(t)p_\Delta(t-0.0002)$ and make use of your graph to find $\int_{-\infty}^{\infty} x(t)\delta(t-0.0002)dt$
   g. Sketch $x(t)p_\Delta(t+0.0002)$ and make use of your graph to find $\int_{-\infty}^{\infty} x(t)\delta(t+0.0002)dt$

10. Generalize on your results in Problem (9) to obtain the value of $\int_{-\infty}^{\infty} x(t)\delta(t-t_o)dt$ for any function x(t)

11. The objective of this problem is to obtain the Fourier Transform of an impulse $\delta(t)$. We begin with a pulse $p(t)$ of area of one as follows



with Fourier Transform as follows

$$X(f) = ha\,\text{sinc}(fa)$$

   a. Describe what happens to the Fourier Transform of the pulse as its pulse width *a* gets smaller. Draw graphs to illustrate. Hint - find where $\text{sinc}(fa) = 0$ as a function of a.
   b. Now make use of our expression for the Fourier Transform as follows

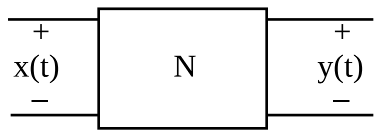
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

to calculate and sketch the Fourier Transform of an impulse $\delta(t)$. Hint - make use of the results from Problem (10) that

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_o)dt = x(t_o)$$

c. Verify that your result in part (b) is consistent with your result in part (a). Then **memorize** the Fourier Transform of an impulse.

12. Given a linear system with transfer function $G(jf)$ as follows

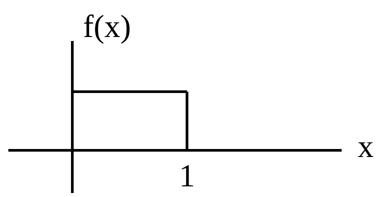


a. Make use of the fact that $F[\delta(t)] = 1$ to show that if $x(t) = \delta(t)$ then

$Y(f)$ = Fourier Transfer of the Impulse Response = $G(jf)$

b. Explain in words why the result in part (a) is true

13. Math Review: Given the following signal



Sketch

a. $f(-x)$
b. $f(-x+2)$
c. $f(-x-2)$