ECE 405 - REVIEW OF THE BASICS - INVESTIGATION 3
REVIEW OF FOURIER SERIES - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we reviewed how to find the Fourier Series of periodic signals in general and pulse trains in particular. The objective of this Investigation is to continue our review of how Fourier Series is used in frequency domain analysis including the calculation of average power.

1. We begin with some review problems. Make use of Euler's Relation as follows

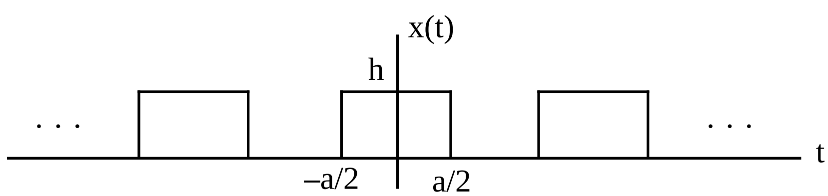
$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta) \qquad r\cos(\theta) = \frac{r}{2}e^{-j\theta} + \frac{r}{2}e^{j\theta}$$

to express the following Fourier Series expansion as a sum of complex exponentials

$$x(t) = 3 + 4\cos(2\pi 1000t + 1.4) + 2.5\cos(2\pi 2000t - 1.3)$$

2. Sketch the double-sided spectral plot of the magnitudes and phases of the harmonics of a periodic signal x(t) with $X_0 = 2$, $X_1 = 3e^{j1.2}$, $X_2 = e^{j}$. Note that $X_{-k} = X_k^*$

3. Given a pulse train as follows



is a sum of complex exponentials with X_k's given by

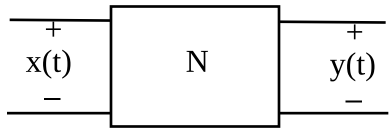
$$X_k = \frac{ha}{T}\frac{\sin(\pi k f_o a)}{\pi k f_o a} = \frac{ha}{T}\text{sinc}(kf_o a) \qquad \text{where} \qquad \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

a. Find the constant X_0 and first five harmonics $X_1, \ldots X_5$ of a pulse train with $h = 5$, $a = 0.0002$ and $T = 0.0015$
b. Make use of your result in part (a) to find $X_{-1}, \ldots X_{-5}$
c. Draw the envelope for our pulse train as given by

$$X_{env}(f) = \frac{ha}{T}\text{sinc}(fa)$$

and then draw in the double-sided spectral plot

4. Now that we know how to find Fourier coefficients X_k of periodic signals $x(t)$ - how to find where in the spectrum a given periodic signal is located - our objective in this and the next problem is to review how they're affected by transfer functions $G(jf)$ of linear circuits like the following



The basic result is that if N is periodic with

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_o t}$$

then by superposition the steady state response of $y(t)$ is given by

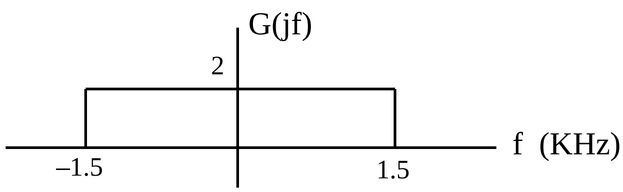
$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi k f_o t} = \sum_{k=-\infty}^{\infty} X_k G(jkf_o) e^{j2\pi k f_o t}$$

Make use of this result to find the steady state response of $y(t)$ as a sum of complex exponentials when the input x(t) is periodic with frequency $f_o = 1000$ Hz and $X_0 = 2$, $X_1 = 3e^{j1.2}$, $X_2 = 3e^{j}$ if $G(jf)$ is given by

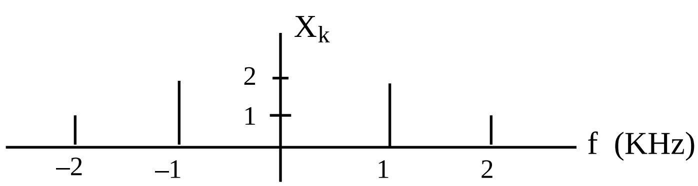
$$G(jf) = \frac{10^4}{10^4 + j2\pi f}$$

We call this way to analyze circuits **frequency domain analysis**. **Memorize** this term.

5. Given a linear circuit with transfer function as follows



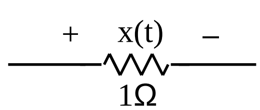
Sketch the spectral plot of the output $y(t)$ if the input $x(t)$ has the following spectral plot



6. The objective of this and the remaining problems in this Investigation is to calculate the average normalized powers of periodic signals $x(t)$ of period T. We call P_N as follows

$$P_N = \frac{1}{T}\int_T x^2(t)dt$$

the average normalized power of x(t) because it is the average power when $x(t)$ is the voltage across a 1Ω resistor as follows



We begin with sinusoids. Show that the average normalized power of the sinusoid $x(t) = A\cos(2\pi ft)$ is P_N as follows

$$P_N = \frac{1}{2}A^2$$

Hint - make use of the fact that $\cos x \cos y = 0.5\cos(x-y) + 0.5\cos(x+y)$

7. Generalizing on the result of Problem (6) we have that if $x(t)$ is periodic with

$$x(t) = c_o + \sum_{k=1}^{\infty} c_k \cos(2\pi k f_o t + \theta_k)$$

then the average normalized power of $x(t)$ can be shown to equal the sum of the average normalized powers of the harmonics as follows

$$P_N = c_o^2 + \sum_{k=1}^{\infty} \frac{c_k^2}{2}$$

Make use of this result to find the average normalized power of

   a. $x(t) = 5 + 3\cos(2\pi 1000t)$
   b. $x(t) = 5 + 3\cos(2\pi 1000t + 1.2)$
   c. $x(t) = 5 + 3\cos(2\pi 1000t) + 2\cos(2\pi 2000t)$

8. Given our result for the average power from Problem (7) as follows

$$P_N = c_o^2 + \sum_{k=1}^{\infty} \frac{c_k^2}{2}$$

   a. Make use of the fact that $c_k = 2|X_k|$ to show that if $x(t)$ is periodic with Fourier coefficients X_k then its average normalized power is can be expressed as follows

$$P_N = c_o^2 + \sum_{k=1}^{\infty} \frac{c_k^2}{2} = |X_0|^2 + 2|X_1|^2 + 2|X_2|^2 + \cdots$$

   with $|X_0|^2$ equal to the normalized power of the DC term and $2|X_k|^2$ equal to the average normalized power of the $k'th$ harmonic.

   b. Then make use of of the fact that $|X_1| = |X_{-1}|$ to show that P_N can be written as follows

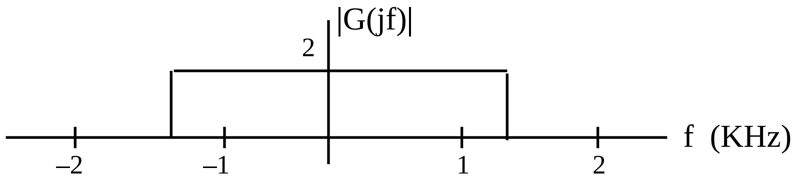
$$P_N = \sum_{k=-\infty}^{\infty} |X_k|^2$$

   Memorize this result.

9. Given a periodic signal x(t) with Fourier Coefficients $X_0 = 2$, $X_1 = 3e^{j1.2}$, $X_2 = e^{j}$

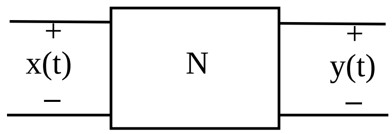
   a. Plot the two-sided **power spectral plot P_N** - the plot of $|X_k|^2$ as a function of k
   b. Find the average normalized power of $x(t)$

10. Find the normalized average power at the output of a filter with the following frequency response



when the input $x(t)$ is periodic with $f_o = 1\,\text{KHz}$ and $X_0 = 2$, $X_1 = 3e^{j1.2}$, $X_2 = e^{j}$

11. Generalizing on the result of Problem (18) we have that if $x(t)$ is the periodic input of a circuit N with frequency response $G(jf)$ as follows

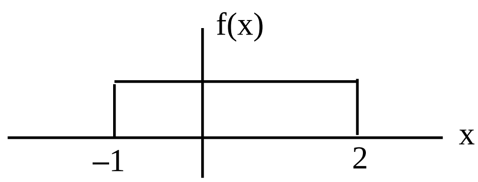


then the average normalized power of y(t) is given by

$$P_N = \sum_{k=-\infty}^{\infty} |Y_k|^2 = \sum_{k=-\infty}^{\infty} |G(jkf_o)X_k|^2 = \sum_{k=-\infty}^{\infty} |G(jkf_o)|^2 |X_k|^2$$

Verify that this expression gives the same result as the one you got in Problem (18)

12. Math Review: Given the following signal



Sketch

a. $2f(x)$
b. $f(x+2)$
c. $f(x-2)$
d. $f(2x)$
e. $f(x/2)$
g. $f(x+2) + f(x) + f(x-2)$