ECE 405 - BASEBAND TRANSMISSION - INVESTIGATION 24
BIT ERROR RATE OF MATCHED FILTERS

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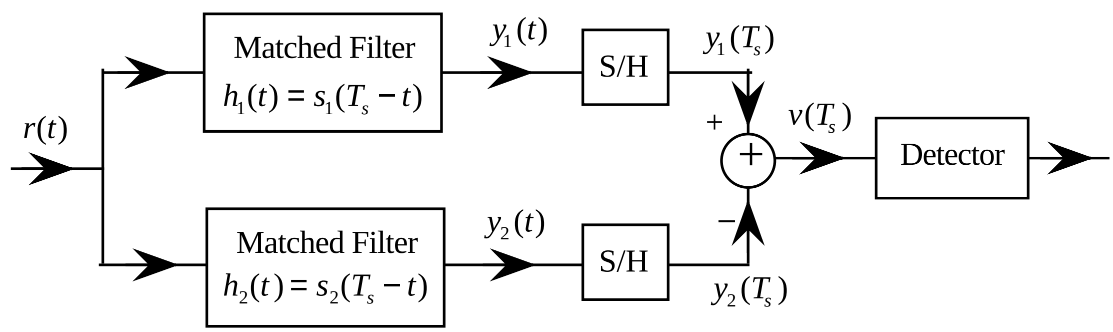
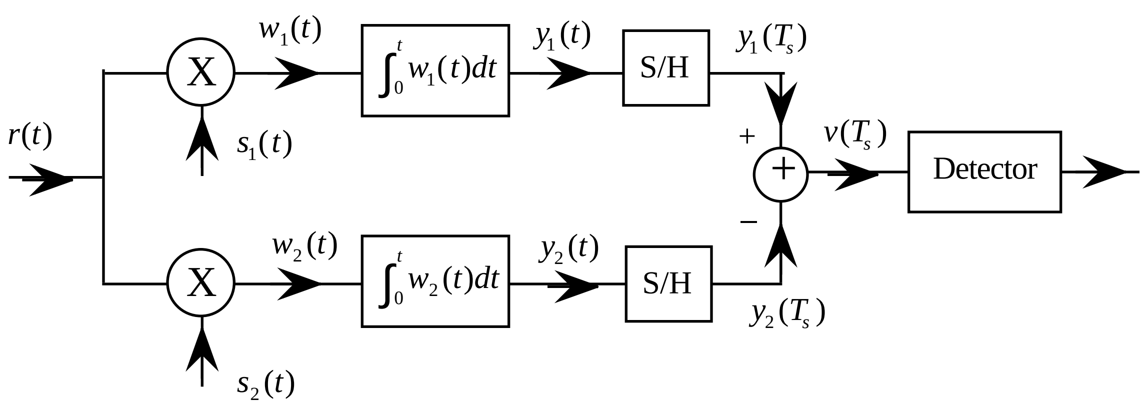
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this Investigation is to make use of the results from the last Investigation to calculate the probability that a signal $s(t)$ will be so distorted by noise that the output of the detector will be in error.

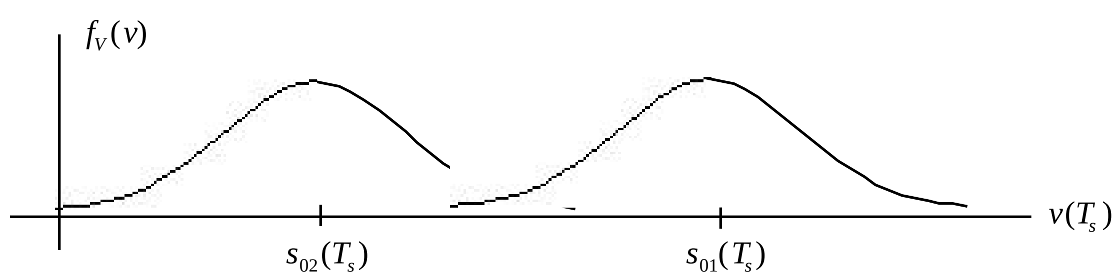
1. From the last Investigation we know that matched filter circuits as follows



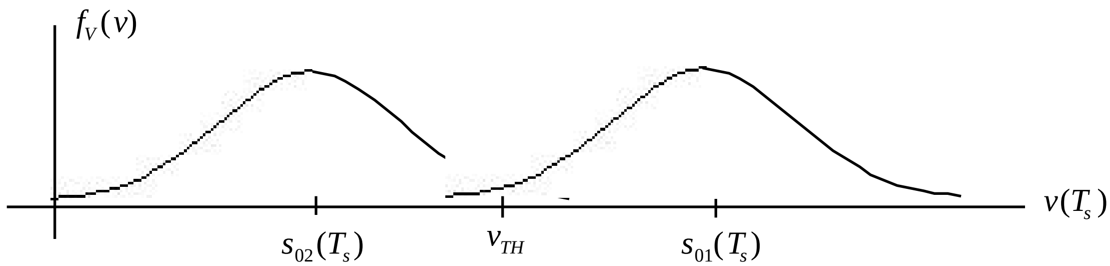
as well as the equivalent correlation circuits as follows



minimize the affects of additive white Gaussian noise. But they don't eliminate it. Therefore $\nu(T)$ at the input to the detector will have a probability density function as follows



when the inputs **$s_1(t)$ for 1 and $s_2(t)$ for 0** are equally likely. The job of the detector is to make its best guess of which signal was sent. This is done by choosing a *threshold voltage* V_{TH} as follows



and then setting the detector *to say* the received signal $r(t)$ is

(1) $s_1(t)$ when $v(T) \geq V_{TH}$
(2) $s_2(t)$ when $v(T) < V_{TH}$.

The value of V_{TH} is chosen to minimize the probability of error. Going through the analysis it can be shown that this will happen where the two probability densities intersect. For equally likely signals like those above V_{TH} is simply the mean as follows

$$\mathrm{V}_{TH} = \frac{s_{01}(T_s) + s_{02}(T_s)}{2}$$

a. Shade in the area for $\mathrm{P(E|1)}$ = Probability of an error when $s_1(t)$ is transmitted
b. Shade in the area for $\mathrm{P(E|0)}$ = Probability of an error when $s_2(t)$ is transmitted
c. What is the area under each of the Gaussian curves if the signals are equally likely
d. Find an expression for P(E) in terms of $\mathrm{P(E|0)}$ and $\mathrm{P(E|1)}$

2. Make use of the following result for Gaussian distributions

$$P(x > b) = \int_b^\infty \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(x-\mu)^2}{2\sigma_n^2}} dx$$

to express P(E) in Problem (1) as a sum of integrals when both $s_1(t)$ and $s_2(t)$ are equally likely. Note that σ_n^2 is equal to the variance of $n_o(t)$ - the noise reaching the detector

3. Make use of the fact that the sum of the two integrals for $P[E]$ in Problem (2) can be written as a single integral as follows

$$P(E) = \int_{V_{TH}}^\infty \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(x-s_{02}(T_s))^2}{2\sigma_n^2}} dx$$

to show that $P[E]$ can be written as follows

$$P(E) = Q\left(\frac{s_{01}(T_s) - s_{02}(T_s)}{2\sigma_n}\right)$$

where $Q(a)$ is given by

$$Q(a) = \int_b^\infty \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma_n^2}}\, dx = \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-z^2/2}\, dz \quad \text{with} \quad a = \frac{b-\mu}{\sigma_n}$$

4. Make use of the following two results

$$s_{01}(T_s) = \int_0^{T_s} s_1(t)\left[s_1(t) - s_2(t)\right]dt = \int_0^{T_s} s_1^2(t)dt - \int_0^{T_s} s_1(t)s_2(t)dt = E_1 - \int_0^{T_s} s_1(t)s_2(t)dt$$

$$s_{02}(T_s) = \int_0^{T_s} s_2(t)\left[s_1(t) - s_2(t)\right]dt = -\int_0^{T_s} s_2^2(t)dt + \int_0^{T_s} s_1(t)s_2(t)dt = -E_2 + \int_0^{T_s} s_1(t)s_2(t)dt$$

to show that $s_{01}(T_s) - s_{02}(T_s)$ in Problem (3) can be expressed in terms of the energies E_1 and E_2 of $s_1(t)$ and $s_2(t)$ is as follows

$$s_{01}(T_s) - s_{02}(T_s) = E_1 - 2\gamma\sqrt{E_1E_2} + E_2$$

where

$$E_1 = \int_0^T s_1^2(t)dt = \text{Energy of } s_1(t)$$

$$E_2 = \int_0^T s_2^2(t)dt = \text{Energy of } s_2(t)$$

$$\gamma = \begin{cases} \dfrac{1}{\sqrt{E_1E_2}}\displaystyle\int_0^T s_1(t)s_2(t)dt & E_1 \neq 0,\ E_2 \neq 0 \\ 0 & E_1 = 0 \text{ or } E_2 = 0 \end{cases} = \text{Correlation coefficient of } s_1(t) \text{ and } s_2(t)$$

5. Now make use of the result in Problem (4) and the fact that the variance of the noise function $n_o(t)$ can be shown to equal

$$\sigma_n^2 = \frac{N_o}{2}\left(E_1 - 2\gamma\sqrt{E_1E_2} + E_2\right)$$

where $N_o/2$ is the power spectral density of the noise $n(t)$ in the channel to show that

$$P(E) = Q\left(\sqrt{\frac{E_1 - 2\gamma\sqrt{E_1E_2} + E_2}{2N_o}}\right)$$

6. Suppose $s_1(t)$ is a "pulse" of amplitude $A = 0$ and duration $T_s = 1$ μsec and $s_2(t)$ is a pulse of amplitude $A = 2$ and duration $T_s = 1$ μsec

   a. Find E_1 for $s_1(t)$
   b. Find E_2 for $s_2(t)$
   c. Find γ
   d. Find the probability of error if $N_o = 10^{-6}$ watts/Hz
   e. Find P(E) in terms of Q as a function of N_o and E_b equal to the average energy of $s_1(t)$ and $s_2(t)$. Hint - express E_1 and E_2 in terms of A and T_s

7. Suppose $s_1(t)$ is a pulse of amplitude $A = 2$ and duration $T_s = 1$ μsec and $s_2(t)$ is a pulse of amplitude $A = -2$ and duration $T_s = 1$ μsec

a. Find E_1 for $s_1(t)$
b. Find E_2 for $s_2(t)$
c. Find γ
d. Find P(E) in terms of Q as a function of N_o and E_b equal to the average energy of $s_1(t)$ and $s_2(t)$
e. How does increasing E_b affect P(E). Explain what's going on

8. The **bit error rate (BER)** tells us the rate at which we can expect errors. For example a bit error rate of 10^{-7} tells us that there will be - on average - one error in every 10^7 bits. Suppose we have a digital communication system transmitting pulses at a rate of 10KBPS = 10^4 bits/sec with $\text{BER} = 10^{-10}$
   a. What's the average number of errors λ per day
   b. What's the average time in days between errors
   c. What's the probability of no errors in a given day assuming the errors have a Poisson distribution with

$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$