ECE 405 - BASEBAND TRANSMISSION - INVESTIGATION 21
INTRODUCTION TO EQUALIZERS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we showed how raised cosine filters can reduce intersymbol interference in communication systems with bandlimited channels. This is all great but in general the receiver still needs to deal with the distortion - especially the phase distortion - caused by the frequency responses $G_C(jf)$ of nonideal communication channels as follows

The objective of this Investigation is to introduce equalizers - filters that "undo" the distortion caused by the channel.

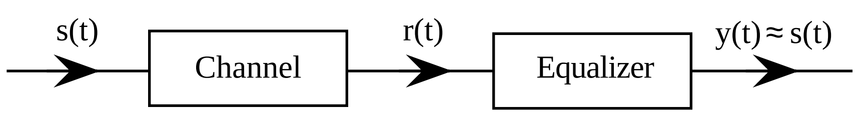
1. As we said in the introduction the nonideal frequency responses $G_C(jf)$ of channels like telephone lines can in general cause distortion in the transmission of signals. The objective of this problem is to illustrate the affects of phase distortion. Suppose in particular that the following channel

causes changes in the phases of the transmitted sinusoids s(t) as follows

$$s(t) = \cos(2\pi 100t) + \cos(2\pi 200t + 0.2)$$
$$r(t) = \cos(2\pi 100t - 0.4) + \cos(2\pi 200t + 0.7)$$

a. Use Mathcad or an equivalent to obtain a graph of the transmitted signal $s(t)$
b. Find the phases $G_C(jf)$ of the channel's frequency response at $\mathrm{f} = 100\,\mathrm{Hz}$ and $\mathrm{f} = 200\,\mathrm{Hz}$
c. Use Mathcad or an equivalent to obtain a graph of the received signal $r(t)$
d. Describe the distortion caused by the phase changes
e. What can you conclude about the affects of phase distortion

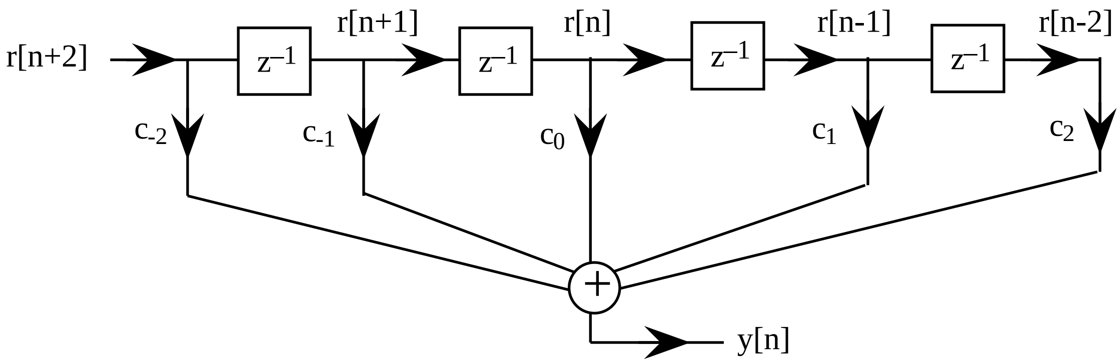
2. In Problem (1) we showed how a channel can distort a signal by changing the phases of its sinusoids. The objective of an **equalizer** in a circuit like the following



is to get us back $s(t)$ from $x(t)$ as much as possible by "undoing" the affects of the channel. Find the phase $G_E(jf)$ of the equalizer at $f = 100$ Hz and at $f = 200$ Hz when

$$s(t) = \cos(2\pi 100t) + \cos(2\pi 200t + 0.2)$$
$$r(t) = \cos(2\pi 100t - 0.4) + \cos(2\pi 200t + 0.7)$$

3. To keep the math and the implementation of equalizer filters as simple and reliable as possible they're usually implemented as FIR (Finite Impulse Response) filters like the following



Note that the c_k' s are the impulse response of the FIR filter

   a. Write the equation for y[n]
   b. Find $y[0]$ if $r[-2]=3$, $r[-1]=2$, $r[0]=0$, $r[1]=-1$, $r[2]=1$ and $c_{-2}=2$, $c_{-1}=1$, $c_0=1.5$, $c_1=1$, $c_2=2$

4. FIR equalizers like those in Problem (3) are great but with $y[n]$ equal to only a finite number of terms $G_E(jf)$ is not going to be able to completely "undo" the phase and magnitude distortion caused by $G_C(jf)$. We need to settle for choosing the coefficients c_k to in some "way" minimize the difference between the transmitted signal $s(t)$ and the output of the equalizer $y(t)$. A common approach is to choose the coefficients c_k of the FIR filter to minimize the expected value of the square of the error as follows

$$\varepsilon = E\left[\left(y(t)-s(t)\right)^2\right]$$

The reason for this choice of error function is that it's relatively straightforward to find its minimum by taking its derivative and setting it to zero.

Make use of the fact that

$$y(t) = \sum_{n=-N}^{N} c_n r\left(t-nT_b\right)$$

to show that setting the derivative of ε to zero as follows

$$\frac{\partial \varepsilon}{\partial c_m} = 0 \qquad m = 0, \pm 1, \ldots, \pm N$$

gives us the following set of **Wiener-Hopf equations** that need to be solved for the coefficients c_k

$$E\left[\left(y(t)-s(t)\right)r\left(t-mT_b\right)\right] = 0 \qquad m = 0, \pm 1, \ldots, \pm N$$

5. The Wiener-Hopf equations in Problem (4) are great except that calculating the expectations in real time can be a hassle - especially for low cost modems that need new coefficients everytime they're dialed. So we need something more practical that gives good results. One alternative - the **Least Mean Squares (LMS) algorithm** developed by Widrow and Hoff - gives us a way to **adaptively** update the coefficients of the FIR equalizer each time the modem is

initialized. The basic idea of the LMS equalizer algorithm is as follows:

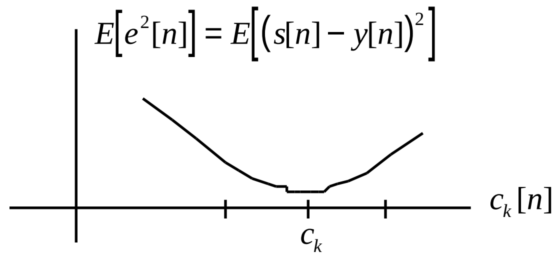
(1) The LMS algorithm starts with initial guesses of $c_k[0]=0$ for each coefficient c_k

(2) The transmitter - before sending any actual data - sends a signal $s[n]$ known to the receiver

(3) The LMS algorithm makes use of the differences between $s[n]$ and $y[n]$ at the output of the equalizer to iteratively (step by step) get as close as possible to the coefficients c_k as follows

$$c_k[0] \rightarrow c_k[1] \rightarrow \cdots \rightarrow c_k[n] \rightarrow c_k[n+1] \rightarrow \cdots c_k$$

that will "as much as possible undo" the distortion caused by the *frequency response* of the channel. In particular the values of the coefficients c_k that minimize the expected value of the error between $s[n]$ and $y[n]$ as indicated in the following graph



(4) The LMS algorithm makes use of an estimate of the slope of $E\left[e^2[n]\right]$ to calculate $c_k[n+1]$ in terms of $c_k[n]$

a. Suppose that after the n'th iteration the slope of $E[e^2]$ is positive. Does this mean that $c_k[n]$ greater than c_k or less than c_k

b. What if the slope in part (a) had been negative

6. Make use of the result in Problem (5) to explain why the following expression for $c_k[n+1]$ in terms of $c_k[n]$

$$c_k[n+1] = c_k[n] - \frac{1}{2}\mu g_k$$

makes sense where

$$g_k = \frac{\partial}{\partial c_k} E\left[e^2[n]\right] = \frac{\partial}{\partial c_k} E\left[(s[n]-y[n])^2\right]$$

is the slope of $E\left[e^2[n]\right]$ at $c_k[n]$ and $\mu > 0$ is a scaling coefficient. Explain in particular the reason for the minus sign. Do this by considering the two cases $g_k > 0$ and $g_k < 0$

7. The key idea that makes the LMS algorithm work is to approximate the expression for g_k as follows

$$g_k = \frac{\partial}{\partial c_k} E\left[e^2[n]\right] = \frac{\partial}{\partial c_k} E\left[(s[n]-y[n])^2\right] = 2E\left[(s[n]-y[n])\frac{\partial}{\partial c_k} y[n]\right] = 2E\left[e[n]r[n-k]\right]$$

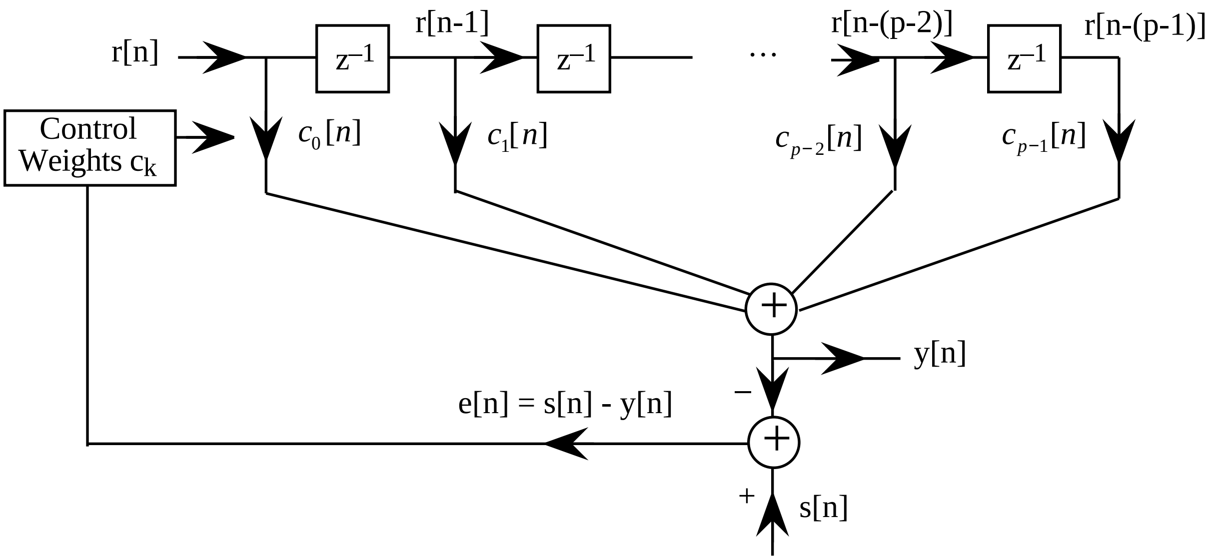
by the following

$$g_k = 2e[n]r[n-k]$$

We therefore have

$$c_k[n+1] = c_k[n] - \frac{1}{2}\mu g_k = c_k[n] - \mu e[n]r[n-k]$$

with $c_k[n+1]$ - the value of the coefficient c_k at the (n+1)st iteration - equal to a function of known parameters $c_k[n]$, $r[n-k]$ and $e[n] = s[n] - r[n]$ as indicated in the following adaptive equalizer



Now to actually implement the LMS algorithm the equalizer needs to know the signal $s[n]$ being transmitted in order to calculate the error $e[n]$. So before sending any real data the transmitter sends a signal $s[n]$ that the equalizer knows the value of so it can calculate $e[n]$. Note that this must be done every time a new connection is made

a. Draw an adaptive FIR equalizer with $p = 3$ for $n = 0$
b. Now make use of the LMS algorithm to find $c_k[1]$ for $k = 0,1,2$ if $p = 3$, $c_k[0] = 0$, s[0] = 1, $r[0] = 0.9$, $r[-1] = 0.1$, $r[-2] = 0$. Use $\mu = 0.2$. Hint - first find y[0] and then e[0] and then substitute into the equation for $c_k[1]$

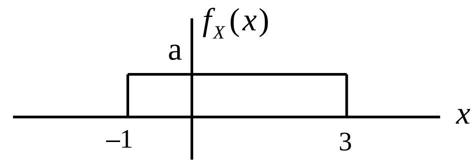
8. Circuit Review - Given a linear circuit
   a. What do we mean by the impulse response $h(t)$ of the circuit
   b. How is $h(t)$ related to the transfer function $G(jf)$ of the circuit

9. Probability Review - Given a continuous random variable X and its probability density function $f_X(x)$
   a. What is $P = \int_a^b f_X(x)dx$ the probability of
   b. What is $\int_{-\infty}^{\infty} x f_X(x)dx$ equal to

10. Probability Review -
    a. What are random variables

b. Give an example of a random variable
c. What information does the discrete probability distribution $f_X(k)$ give us when $k = 1$

11. Probability Review - Given the following continuous uniform probability density



Find each of the following

a. a
b. $P(1 \le x \le 2)$
c. $P(x \ge 1)$