ECE 405 - TRANSITION TO DIGITAL COMMUNICATION - INV 13
INTRODUCTION TO SAMPLING - PART I

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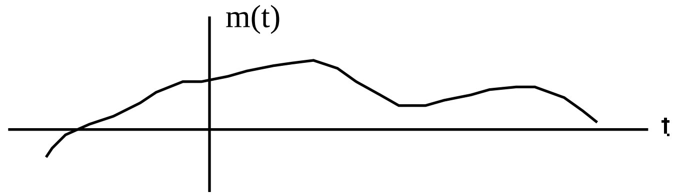
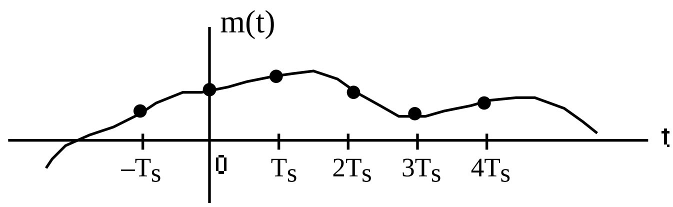
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

Up to now we've been transmitting message signals $m(t)$ like the following

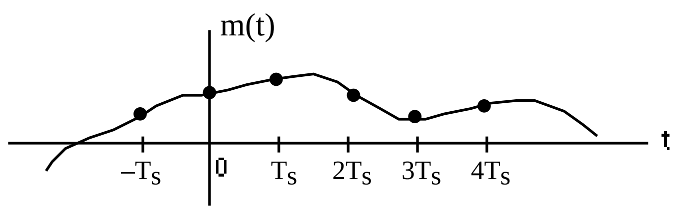


with analog communication systems like AM and FM that transmit the "whole" signal. But in digital communications we only transmit the values of $m(t)$ at the *sample times* nT_s as indicated in the following graph



The objective of this Investigation is to see how fast we have to sample $m(t)$ for the receiver to be able to reconstruct $m(t)$ from its samples $m[n] = m(nT_s)$.

1. When the time T_S between samples is always the same as follows



then we call it **uniform sampling**. We will always mean uniform sampling when we say sampling in this class. Given the following message signal

$$m(t) = 5\cos(2\pi 1000t)$$

   a. Find the first five samples of $m(t)$ for $T_s = 0.4\text{msec}$. Start at time $t = 0$. Put your results in a Table
   b. Draw your samples on a graph of the sinusoid

2. The **sampling frequency f_S** is equal to the number of samples/sec
   a. Find f_s as a function of T_s
   b. How does increasing T_s affect f_s

c. Find the first five samples of $m(t) = 5\cos(2\pi 1000t)$ starting at time $t = 0$ with $f_s = 2.5$ KHz. Put your results in a Table.

3. As we see from Problems (1) and (2) it's straightforward to obtain a signal's samples. But to reconstruct a signal from its samples requires more thought. The problem is that in general there's an infinite number of signals that go through any given set of samples. Illustrate this fact by drawing three signals that have the same four sample values.

4. From Problem (3) we now know that we need more information about m(t) before we can reconstruct it from its sample values $m[n] = m(nT_s)$. The sample values alone are not enough. The key insight comes from the observation that if m(t) is not changing "too quickly" between samples then we can get a good approximation of m(t) by simply "connecting the dots".

   Now for $m(t)$ to not be "changing too quickly" between samples means there must be limitations on its spectrum. What this all leads to is what we call the **Sampling Theorem** as follows

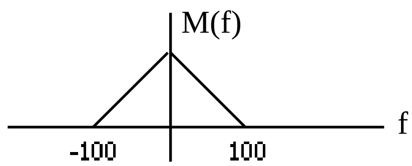
   A signal $m(t)$ can be reconstructed from its samples $m[n] = m(nT_s)$ if

   (1) $m(t)$ is bandlimited by a frequency f_b
   (2) The sampling frequency f_s is large enough so that $f_s > 2f_b$

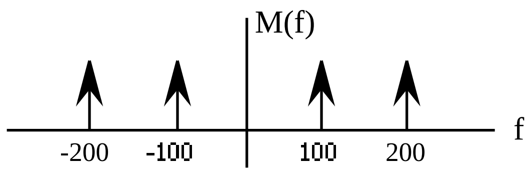
   Note that we call $2f_b$ of a bandlimited signal its **Nyquist frequency**.

   Memorize the Sampling Theorem. And then find the Nyquist frequency for each of the following signals

   a. $m(t) = 5\cos(2\pi 1000t)$
   b. $m(t) = 5\cos(2\pi 1000t) + 3\cos(2\pi 2000t)$
   c. $m(t)$ with the spectrum



   d. $m(t)$ with the spectrum



5. The trick to deriving the Sampling Theorem is to express the sample values $m[n]$ of a signal $m(t)$ in an *equivalent* analog form as follows

$$m_\delta(t) = \cdots + m[-1]\delta(t + T_s) + m[0]\delta(t) + m[1]\delta(t - T_s) + m[2]\delta(t - 2T_s) + \cdots$$

   which is zero everyplace except at the sampling times $t = \cdots, -T_s, 0, T_s, 2T_s, \cdots$

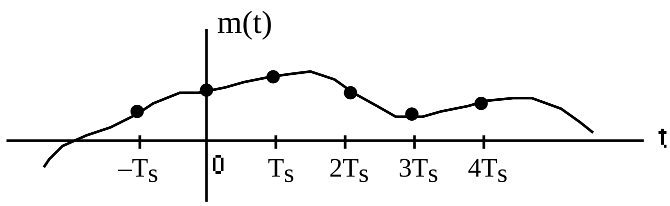
   a. Find and sketch $m_\delta(t)$ for the samples

$$m[-2] = 1 \qquad m[-1] = -2 \qquad m[0] = 2 \qquad m[1] = 3 \qquad m[2] = 1$$

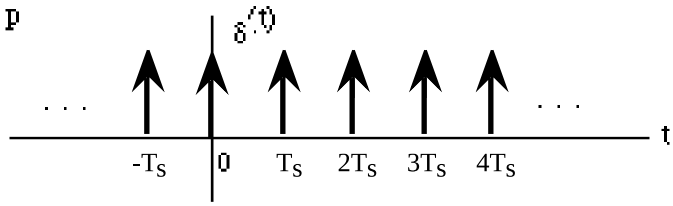
b. Find the samples $m[n]$ of $m(t)$ with impulse sampled signal $m_\delta(t)$ as follows

$$m_\delta(t) = 2\delta\left(t+T_s\right) + \delta\left(t\right) + -2\delta\left(t-T_s\right) + 2\delta\left(t-2T_s\right)$$

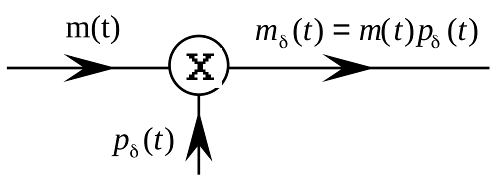
6. Upon examination of the results from Problem (5) we see that we can obtain $m_\delta(t)$ for a given signal $m(t)$ by simply multiplying $x(t)$



by an impulse train $p_\delta(t)$ as follows



We refer to this as **ideal or impulse sampling** and represent it as follows

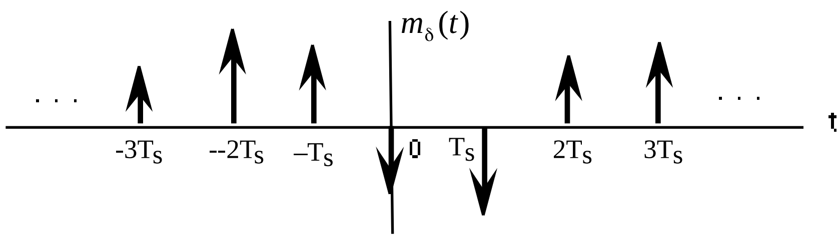


with a block diagram as follows

Now suppose we impulse sample the signal $m(t) = \cos(2\pi 100 t)$ with $f_s = 300$ samples/sec

a. Find an expression for $m_\delta(t)$ as a sum of impulses
b. Then sketch $m_\delta(t)$ for -15 msec $\le$ t $\le$ 15 msec

7. The objective of this problem is to sketch the spectrums $M_\delta(f)$ of impulse sampled signals $m_\delta(t)$ like the following



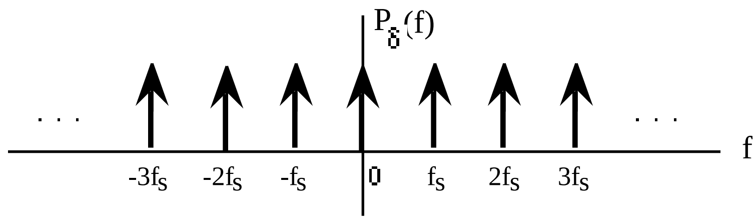
with

$$m_\delta(t) = m(t)p_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

The trick is to make use of the fact that the Fourier Transform of a product is equal to the convolution of the Fourier Transforms as follows

$$M_\delta(f) = F[m(t)p_\delta(t)] = M(f) \quad P_\delta(f)$$

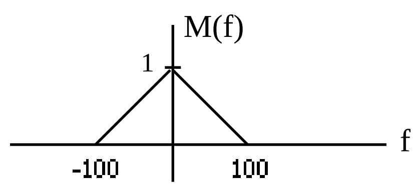
where $P_\delta(f)$ - the Fourier Transform of the pulse train of impulses $p_\delta(t)$ - is itself a pulse train of impulses as follows



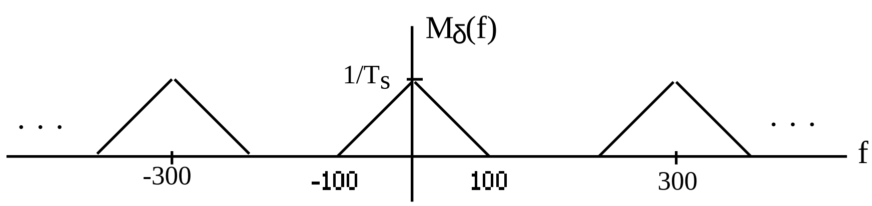
with areas equal to $1/T_s$ and frequency equal to $f_s = 1/T_s$. If we now carry out the convolution we get

$$M_\delta(f) = F[m(t)p_\delta(t)] = M(f) \quad P_\delta(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} M(f - kf_s)$$

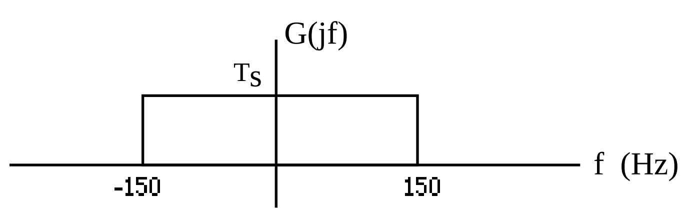
Memorize this result. And then make use of it to sketch $M_\delta(f)$ for the signal m(t) with the following spectrum with f_S = 300 samples/sec



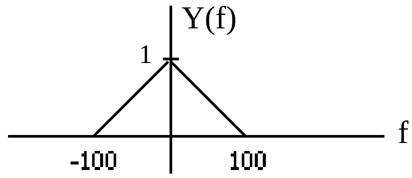
8. If we now take the result from Problem (7) as follows



we see that if we pass $m_\delta(t)$ through an ideal lowpass filter with cutoff frequency $f_c = f_s/2 = 150\,Hz$ as follows

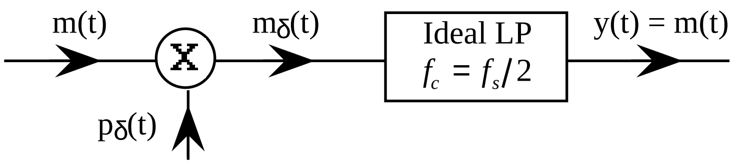


then we'll end up with a signal $y(t)$ having the following spectrum

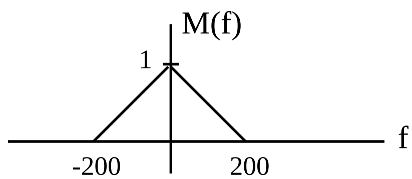


But this is the same spectrum as that of $m(t)$ and so $y(t) = m(t)$

Now pulling everything together we see that we can sample and then recover a bandlimited signal $x(t)$ by impulse sampling and then lowpass filtering it as follows



as long as $f_s > 2f_b$. Let's suppose, in particular, that $x(t)$ has the following spectrum



a. What would you choose for f_s
b. Sketch $M_\delta(f)$ for your f_s
c. Sketch the lowpass filter for recovering $m(t)$
d. Sketch $Y(f)$
e. What's the relation between your $M(f)$ and $Y(f)$ and therefore $m(t)$ and $y(t)$