ECE 405 - ANALOG COMMUNICATIONS - INVESTIGATION 11
INTRODUCTION TO POWER SPECTRAL DENSITIES

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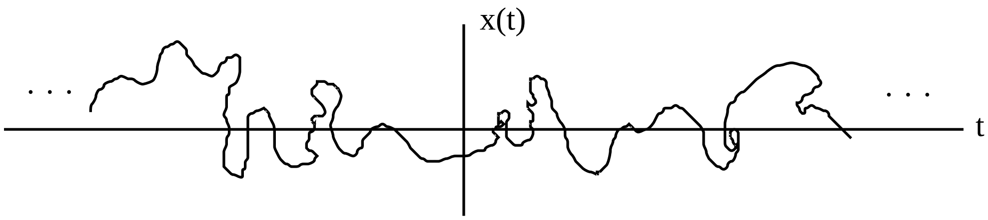
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In previous Investigations we made use of Fourier Series and Fourier Transforms to calculate the spectrums bandwidths of AM and FM signals. This works great for "nice" signals like periodic pulse trains and nonperiodic single pulses. But when we try to find the spectrums and bandwidths of general message signals and randomly varying noise the Fourier Series and Fourier Transform don't work "very well". The objective of this Investigation is to introduce how we can get around this problem with power spectral densities.

1. As we know from previous Investigations the Fourier Transform works great for signals like pulses and modulated pulses. But it can be shown that the Fourier Transform integral as follows

$$\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}dt$$

does not in general converge and so does not exist for general nonperiodic signals like the following



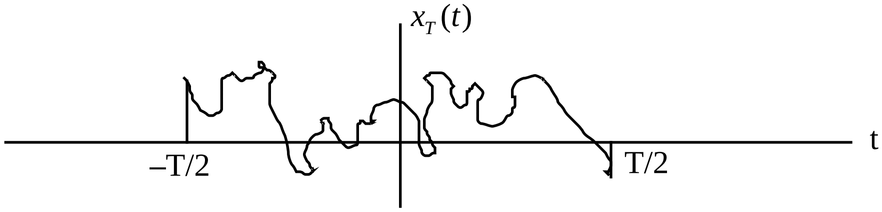
that are not *square integrable* as follows

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

a Show that $x(t) = e^t u(t)$ is not square integrable and therefore does not have a Fourier Transform

b. Find your own example of a signal that is not square integrable

2. One way to find the "bandwidths" of signals like those in Problem (1) that are not square integrable is to take the Fourier Transforms of "representative" truncated samples $x_T(t)$ like the following



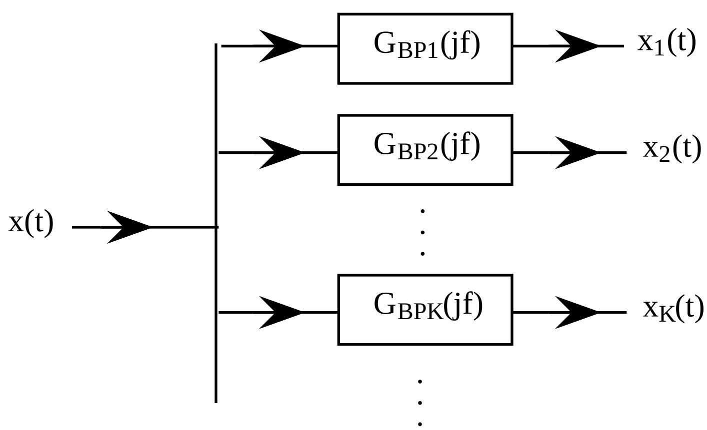
But this takes special care and fine tuning. On the other hand we can calculate the *power* *spectral densities* of deterministic signals $x(t)$ as long as $x(t)$ is a **power signal** - has finite average power as follows

$$0 < P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} p(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt < \infty$$

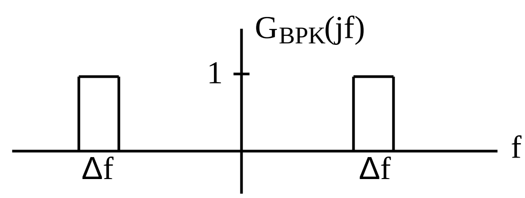
a. Explain why P_{av} is the average *normalized power* - the average rate at which energy is dissipated in a 1Ω resistor with voltage $x(t)$
b. Show that $x(t) = A\cos(2\pi bt)$ is a power signal by finding the corresponding P_{av}

3. The objective of this and the next problem is to define **power spectral density** $S_x(f)$ introduced in Problem (2). If $x(t)$ is a power signal then we can obtain its power spectral density as follows

(1) Put $x(t)$ through a bank of bandpass filters as follows



with bandwidths equal to Δf as follows



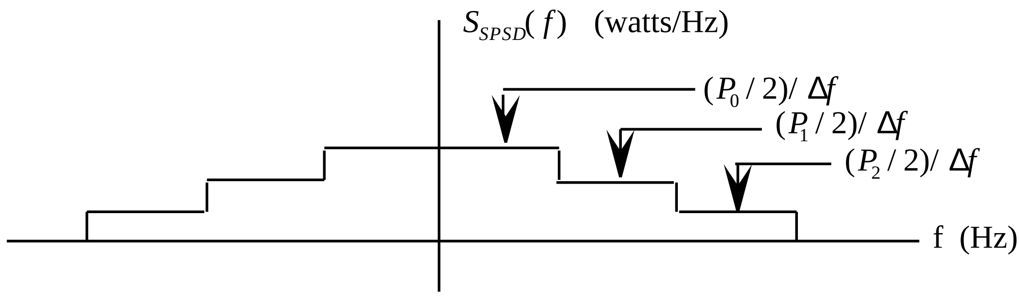
(2) Find the average power P_k at the output of each of the filters as follows

$$P_k = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_k^2(t)dt$$

(3) Find the power spectral density in each interval as follows

$$S_{PSD}(k) = \frac{P_k}{\Delta f} = \frac{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_k^2(t)dt}{\Delta f}$$

(4) Combine the power spectral densities to form a *double-sided staircase power spectral density* $S_{SPSD}(f)$ as follows

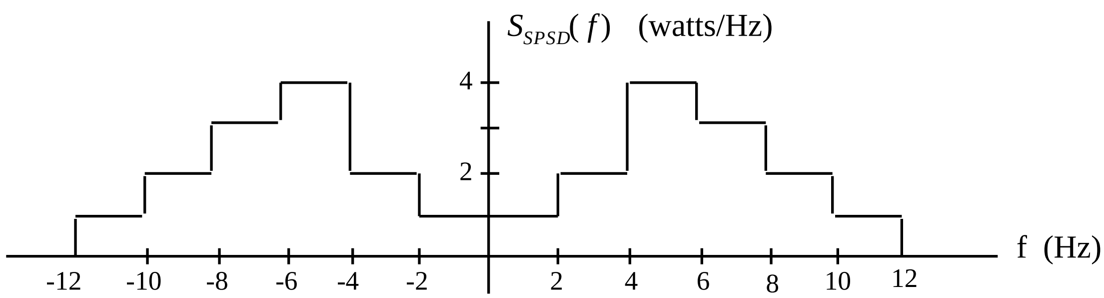


a. Find and sketch the staircase power spectral density $S_{SPSD}(f)$ of a signal $x(t)$ with $P_0 = 5$ watts, $P_1 = 3$ watts, $P_2 = 2$ watts and $\Delta f = 2$ Hz

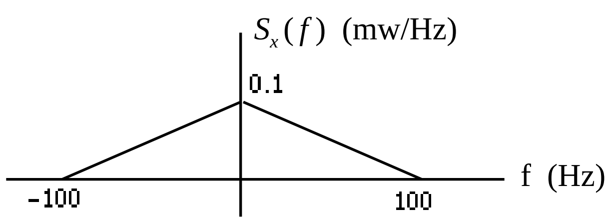
b. Now by superposition the total average power is the sum of all the average powers of $x_1(t),\ x_2(t), \ldots$ and so is equal to the integral of the staircase power spectral density as follows

$$P_{av} = \int_{-\infty}^{\infty} S_{SPSD}(f)df$$

Make use of this result to find the total average power P_{av} of a signal with the following power spectral density

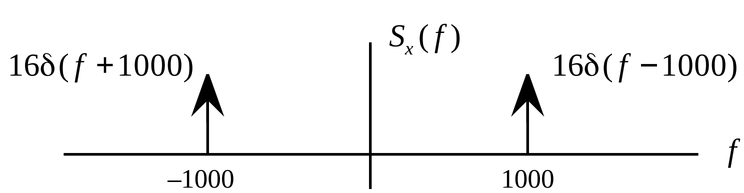


4. As we make the passbands Δf of the bandpass filters narrower and narrower the staircase power spectral density plots metamorphize into nice smooth power spectral densities $S_x(f)$ like the following



Find the total average power P_{av} of the signal $x(t)$ with this power spectral density.

5. Explain why the power spectral density of the sinusoid $x(t) = 8\cos(2\pi 1000t)$ is as follows



6. Power spectral densities are particularly nice not only because they are straightforward to measure but also because it's straightforward to obtain closed form expressions for them in terms of their *autocorrelations*

   To actually calculate the power spectral density of a power signal $x(t)$ we begin with the fact that its average power P_{av} is equal to the following limit

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)\,dt$$

   where $x_T(t)$ is a truncated $x(t)$. But $x_T(t)$ is an **energy signal** - a signal that delivers a finite amount of energy as follows

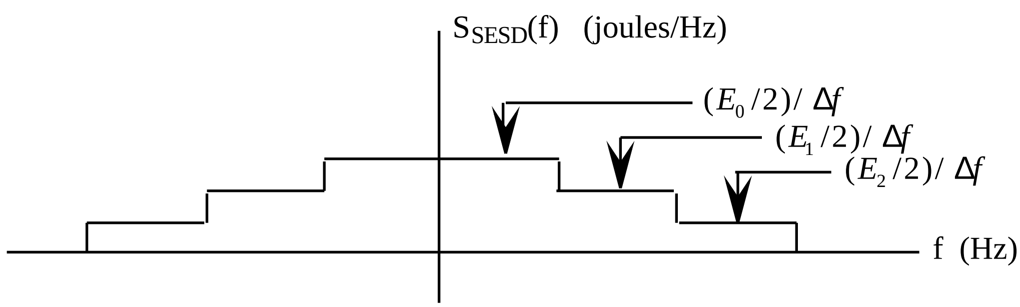
$$E = \int_{-\infty}^{\infty} p(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

   to a 1Ω resistor. The objective of this and the next several problems is to develop what we mean by *energy spectral density* in order to lay the foundation for calculating the power spectral density in a later Investigation

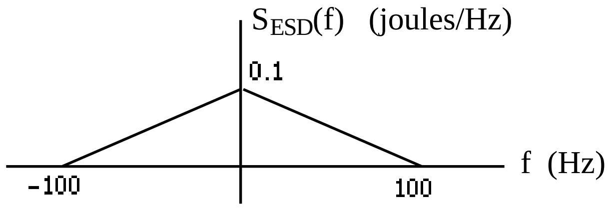
   a. Explain why energy signals are not power signals
   b. Verify that the following pulse is an energy signal

   c. Verify that the unit step function u(t) is not an energy signal
   d. Come up with your own example of an energy signal

7. The **energy spectral density** of an energy signal is basically the same as the power spectral density of a power signal except that it gives us the energy density as a function of frequency rather than the average power as a function of frequency. We begin, as we did for the power spectral density, by passing the signal through a bank of bandpass filters to obtain energies E_k which we then make use of to obtain two-sided staircase energy spectral densities as follows



   Now taking the limit as $\Delta f \to 0$ just like we did in the case of power signals we obtain energy spectral density plots like the following

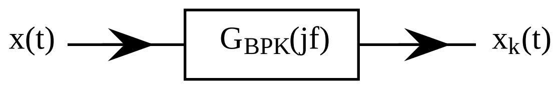


Find the total energy of the signal x(t) with this energy spectral density.

8. The objective of this Problem is to make use of *Parseval's Theorem* for energy signals $x(t)$ as follows

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

to obtain an expression for the energy spectral density of $x(t)$ in terms of its Fourier Transform $X(f)$. From Parseval's Theorem we see that the energy of $x_k(t)$ at the output of the k'th bandpass filter as follows



is given by

$$E_k = \int_{k\Delta f}^{(k+1)\Delta f} |X(f)|^2 df$$

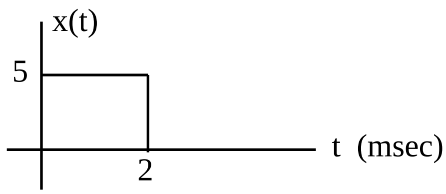
and so the energy spectral density in this interval is given by the following expression

$$\frac{E_k}{\Delta f} = \frac{\int_{k\Delta f}^{(k+1)\Delta f} |X(f)|^2 df}{\Delta f} \approx \frac{|X(f_k)|^2 \Delta f}{\Delta f} = |X(f_k)|^2$$

when Δf is small. So in the limit as $\Delta f \to 0$ we see that the energy spectral density of an energy signal $x(t)$ is equal to the square of the magnitude of its Fourier Transform as follows

$$E_{ESD}(f) = |X(f)|^2$$

Make use of this result to find and sketch the energy spectral density of the following pulse



In a later Investigation we'll extend our results above and show how both energy and power spectral densities can be calculated from their autocorrelations.