ECE 315 - DISCRETE RANDOM VARIABLES - INVEST 9
POISSON EXPERIMENTS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From our previous Investigations we know how to calculate and make use of the probability distribution functions of random variables for Bernoulli experiments like the flipping of coins. The objective of this Investigation is to introduce Poisson experiments. Poisson experiments are in some ways similar to Bernoulli experiments but are fundamentally different. We introduce the properties of Poisson experiments with an example from the telephone industry.

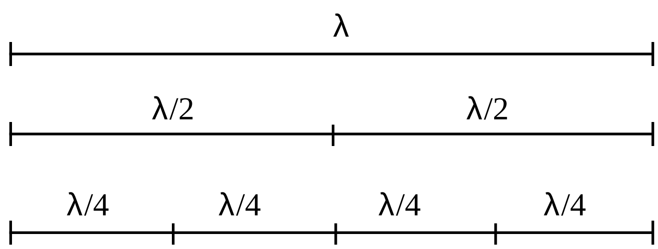
1. When we examine telephone calls we observe the following:

    (1) The average number of calls λ in any time T is proportional to T
    (2) The number of calls in any two non-overlapping time intervals is independent
    (3) No more than one call can arrive at any given time - there is always some time between any two calls

    It is these properties that characterize **Poisson experiments**. The **Poisson distribution** $f_X(k)$ for arriving telephone calls is then equal to the probability of k calls in time T.

    Note in particular how similar the expectations of the binomial and Poisson are: the expectation of a binomial random variable is proportional to the number of trials with $E[X] = np$ while the expectation of a Poisson distribution is proportional to the length of the interval T

    Now as a result of Property (1) above we see that when we divide the time interval T into successively smaller subintervals as follows



    then the average number of phone calls in each subinterval is going to get smaller and smaller as indicated. If we continue this subdividing until the number of subintervals n is so large that $\lambda/n << 1$ then virtually every subinterval will contain either 0 or 1 calls. Which means that each subinterval can be approximated by a Bernoulli trial with two possible outcomes. Which means that if n is large we can approximate the probability $f_X(k)$ of k calls in a time interval T of a Poisson distribution by the probability of k successes in n trials of a binomial distribution as follows

$$f_X(k) \approx \binom{n}{k} p^k (1-p)^{n-k}$$

    a. Rewrite the expression for $f_X(k)$ with p replaced by a function of λ and n. Hint - make use of the fact that for binomial distributions $\lambda = np$

b. Why do you think that the Poisson distribution is applicable in those cases where successes occur "completely at random" - where a success is equally likely to occur in any two intervals of the same length

2. Given our binomial approximation to a Poisson distribution as follows

$$f_X(k) \approx \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} (\lambda/n)^k \left(1-(\lambda/n)\right)^{n-k}$$

   a. Calculate $f_X(1)$ for one minute using n = 10, 50 and 100 subintervals if the average number of phone calls per minute is $\lambda = 1.5$
   b. What's happening to the values of the binomial approximation of the Poisson distribution in part (a) as n increases

3. If we now take the limit as $n \to \infty$ then our binomial approximation for $f_X(k)$

$$f_X(k) \approx \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} (\lambda/n)^k \left(1-(\lambda/n)\right)^{n-k}$$

   will metamorphize into the following expression

$$\boxed{f_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

   for the **Poisson** distribution equal to the probability of k phone calls during an interval with an average of λ phone calls. **Memorize** this expression. Then

   a. Make use of this exact expression for $f_X(k)$ to find $f_X(1)$ in Problem (2).
   b. How close is your result to the approximations of Problem (2). Put the results in a Table

4. Make use of the Taylor Series approximation

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

   to confirm that the average number of phone calls per minute for the Poisson distribution

$$E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

   is in fact equal to λ as claimed. Hint - write out the first couple of terms of the sum as follows

$$E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \left(1\frac{\lambda}{1!} + 2\frac{\lambda^2}{2!} + 3\frac{\lambda^3}{3!} + \cdots\right) e^{-\lambda}$$

   and then factor out λ

5. Suppose a phone we are monitoring receives an average of 3 phone calls/hour
   a. What's the probability of exactly two phone calls in the hour
   b. What's the probability of at least two phone calls in the hour

6. Suppose a given phone receives an average of 10 calls in an hour.
    a. What will be the average number of calls it will receive in 2 hours
    b. What will be the average number of calls it will receive in 15 minutes

7. Suppose a given phone receives an average of 3 calls/hour
    a. What's the probability it receives two calls from 1pm to 2pm
    b. What's the probability it receives two calls from 1pm to 2pm and two calls from 2pm to 3pm
    c. What's the probability it receives exactly four calls from 1pm to 3pm
    d. Why is the probability in part (c) larger than the probability in part (b)
    e. What's the probability it recieves exactly four calls from 1pm to 4pm with two of them from 1pm to 2pm and the other two from 3pm to 4pm

8. Find and plot the probability distribution functions $f_X(k)$ as a function of k for k=0 to k=10 for the Poisson distribution with
    a. $\lambda = 0.5$
    b. $\lambda = 5$
    c. Describe how the graphs are similar and how different
    d. Verify that the maximums are around $k = \lambda$

9. Given the following general expression for the variance of a random variable X

$$Var[X] = E\left[(X-\mu)^2\right] = \sum_{k=0}^{\infty}\left(k - E[X]\right)^2 f_X(k)$$

    a. Write out the equation for Var[X] when X is Poisson with $\lambda = 2$
    b. Approximate the infinite sum in part (a) by the sum of the first ten terms. You might find Mathcad helpful here.
    c. Verify that your approximation for the variance of X in part (b) is close to the exact value - which can be shown to be λ. **Memorize** this result

10. We started out this Investigation by approximating the Poisson distribution by the binomial approximation. So clearly we should be able to approximate the binomial by the Poisson when n is large (n > 100) and $p = \lambda/n$ is small ($p < 0.01$). Make use of this result to find the probability that exactly 3 resistors in a lot of 5000 will be defective if the probability of any given resistor being defective is 0.001.