ECE 315 - DISCRETE RANDOM VARIABLES - INVEST 7
EXPECTATIONS OF DISCRETE RANDOM VARIABLES

WINTER 2004 **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this investigation is to calculate the expectations or averages of random variables. Expectations are numbers that help us characterize and compare probability distributions just like time constants and 3dB frequencies help us characterize and compare linear circuits.

1. We begin with a review problem. Suppose we flip a fair coin three times. Find
   a. $f_X(2)$ if X is the binomial random variable
   b. $f_X(2)$ if X is the geometric random variable

2. The objective of this problem is to calculate some averages
   a. What's the average value of the sequence of numbers 2 3 2 4 4 3
   b. What's the average value of a sequence of n numbers of which 1/4 are 2's, 1/3 are 3's and 5/12 are 4's
   c. What is the average value of the random variable X if the results of doing the random experiment a whole bunch of times are as follows 2 2 3 2 4 3 2
   d. What's the average value of the random variable X with values 2, 3 and 4 if $f_X(2) = 1/4$, $f_X(3) = 1/3$, $f_X(4) = 5/12$. Explain how you got your answer.

3. In Problem (2) we went through a series of calculations of averages that ended up in parts (c) and (d) with us calculating the average value of random variables X. We refer to the average value or **mean** of a random variable X as its **expectation** or **expected value** $\mu = E[X]$. Find the mean or expected value of the random variable X with values $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$ if
   a. After doing the random experiment a whole bunch of times we obtain the data 1 1 2 4 1 3
   b. $f_X(1) = 1/6$, $f_X(2) = 1/3$, $f_X(3) = 1/5$, $f_X(4) = 3/10$
   c. The probability distribution is uniform

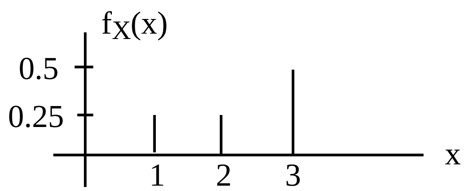
4. Generalize on your results in Problems (21) and (3) to come up with an expression for the expectation $\mu = E[X]$ of a random variable X with values x_1, x_2 and x_3 having probabilities $f_X(x_1)$, $f_X(x_2)$ and $f_X(x_3)$.

5. Generalizing on the results of Problem (4) we have that the expected value of a discrete random variable X with values $x_1, x_2, \ldots, x_n$ and probability distribution function $f_X(x)$ is given by the following expression

$$\mu = E(X) = \sum_{i=1}^{n} x_i f_X(x_i)$$

   Describe in words why this expression gives us the expectation of X. Be sure to note that the values of the random variables are the values of the x_i's and not the values of the integers 1, 2, . . . , n. **Memorize** this expression

6. Make use of the expression in Problem (5) to find the expected value of the random variable X with probability distribution function as follows



7. The objective of this and the next problem is to find the expectation of a binomial random variable. We begin with a binomial random variable X with n = 1 and probability of success p
   a. Find the probability distribution function of X
   b. Make use of your result in part (a) to show that E[X] = p

8. Generalizing on the result of Problem (7) it can be shown that the expectation of a general binomial random variable is $E[X] = np$. The objective of this problem is to verify this result for the case n = 3 and p = 1/2 for k = 0, 1, 2, 3
   a. Make use of the binomial probability distribution function $f_X(k) = \binom{n}{k} p^k q^{n-k}$ to find the probability distribution function of X
   b. Make use of the equation from Problem (5) to find E[X] as follows

$$\mu = E(X) = \sum_{i=1}^{n} x_i f_X(x_i)$$

   c. Verify that your result in part (b) agrees with the fact that E[X] = np for binomial random variables - as we will show later. **Memorize** this result
   d. The objective of this part of the problem is to experimentally test your result in part (b). Flip three coins twenty-five times and then calculate the mean number of heads
   e. Now compare your calculated and measured results in a Table as follows

Calculated E[X]	Measured E[X]	% Difference

   How close are your calculated and measured results

9. The objective of this problem is to experimentally estimate the expected value of a geometric random variable
   a. Do a coin flipping experiment with a fair coin to estimate the expected value of a geometric random variable - the average number of times we need to flip a coin until we get a head
   b. How close is your result in part (a) to the theoretical value of 1/p. Put your results in a Table

10. The objective of this and the next two problems is to find the expectations of functions of random variables. Suppose, in particular, that the random variable X has the following probability distribution function

$$f_X(1) = 1/3 \qquad f_X(2) = 1/2 \qquad f_X(3) = 1/6$$

   a. Find E[X]

b. Now find the probability distribution function $f_Y(y)$ of the random variable $Y = 2X$. Set up a Table with columns for X, Y and $f_Y(y)$
c. Make use of your result in part (b) to find $E[Y] = E[2X]$
d. Find the probability distribution function $f_W(w)$ of the random variable $W = 3X$

11. Given the same random variable X as in Problem (9) with probability distribution function

$$f_X(1) = 1/3 \qquad f_X(2) = 1/2 \qquad f_X(3) = 1/6$$

Now suppose $Y = 2X^2$
   a. Find the probability distribution function $f_Y(y)$ of Y
   b. Make use of your result in part (a) to find E[Y]

12. Generalizing on the expression in Problem (5) we have that the expectation of $Y = g(X)$ is given by

$$E[Y] = E[g(X)] = \sum_{i=1}^{n} g(x_i)\, f_X(x_i)$$

Memorize this result. Then
   a. Describe in words the above expression for E[Y]
   b. Write out the sum for n=3
   c. Use this expression for E[Y] to calculate the expectation of $Y = 2X^2$ for $f_X(x)$ from Problem (10) as follows

$$f_X(1) = 1/3 \qquad f_X(2) = 1/2 \qquad f_X(3) = 1/6$$

   Hint - set up and make use of a Table as follows

x	g(x)	$f_X(x)$

   d. Verify that your result in part (c) agrees with your result in Problem (10). Put your results in a Table

13. Calculate $E\left[(X-1)^2\right]$ for X with the following probability distribution function

$$f_X(1) = 1/3 \qquad f_X(2) = 1/2 \qquad f_X(3) = 1/6$$

14. Make use of our general relation

$$E[Y] = E[g(X)] = \sum_{i=1}^{n} g(x_i)\, f_X(x_i)$$

to show that $E[kX] = k\,E[X]$

15. The objective of this problem is to illustrate how to calculate conditional expectation as follows

$$E[X|Y = y] = \sum_{x} x f_{X|Y}(x|Y = y)$$

Calculate E[X|Y=0] if X is the number of heads when we flip a fair coin three times and Y = 0 means the number of heads is even. Note that 0 is an even number