ECE 315 - DISCRETE RANDOM VARIABLES - INVEST 6
BERNOULLI EXPERIMENTS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last investigation we saw that random variables are functions that assign numerical values to the outcomes of random experiments. The objective of this investigation is to introduce a special class of random experiments called Bernoulli experiments. And then define some random variables for them. Bernoulli experiments are of particular interest not only because they're so common but also because so many random experiments can be approximated by them.

1. We begin with a review problem. Find two random variables X and Y that assign numerical values to the outcomes when we flip a coin three times.

2. The objective of this problem is to define what we mean by a trial. When we perform a random experiment like flipping a coin three times then we refer to each flip as a **trial**. And we refer to the experiment of flipping the coin three times as a **sequence of three trials**. Make up your own example of a random experiment that is a sequence of trials.

3. **Bernoulli trials** are a special kind of trial having the following properties

   (1) Each trial has one of two possible outcomes which we refer to as *success* and *failure* (like heads and not heads)
   (2) The probability p of success and the probability $q = 1 - p$ of failure is the same for each trial
   (3) The trials are independent

   Given all this **Bernoulli experiments** are random experiments consisting of sequences of Bernoulli trials. **Memorize** the properties of Bernoulli experiments. Then make up your own example of a Bernoulli experiment. What are the trials of your experiment.

4. Now let us consider the simple Bernoulli experiment of flipping a fair coin (p = q = 1/2) three times.
   a. First justify that this random experiment is in fact Bernoulli
   b. What is the probability of the outcome HTH in this Bernoulli experiment

5. Generalizing on the example of the last problem we have that the probability of a given outcome of a Bernoulli experiment like

   SSFSFF

   where the S's are successes and the F's failures is - since the trials are independent - simply the product of the corresponding probabilities as follows

   P[SSFSFF] = ppqpqq

   where p is the probability of success and q the probability of failure. Make use of this result to find the probability of the following outcome when a coin is flipped 8 times with P(H) = 0.6

   HHTTHTHH

6. In the last several problems we defined and gave several examples of Bernoulli experiments. The objective of this and the rest of the problems of this Investigation is to define some random variables on Bernoulli experiments. The **binomial random variable** of a Bernoulli experiment is simply the number of successful trials.
   a. Suppose, in particular, that we flip a fair coin a total of three times. Set up a Table of the possible outcomes and the corresponding values of the binomial random variable function if success is a head. Then calculate the values of the binomial probability distribution function.
   b. Generalizing on the result of part (a) it can be shown that the probability distribution function for the binomial random variable is given by

$$f_X(k) = \binom{n}{k} p^k q^{n-k}$$

   where

   n = number of trials
   k = number of successes
   p = probability of a success
   q = 1 – p = probability of a failure
   $\binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$ = number of combinations of n things taken k at a time

   Use this equation to check your results in part (a)
   c. Suppose that the probability of any given scope being in calibration after a month of use is p = 0.6. What's the probability that exactly two scopes from a lab of four will be in calibration after a month
   d. What's the probability that at least two of the scopes in part (c) will still be in calibration after a month of use. Explain how you got your answer

7. Calculate and then make a plot of the binomial probability distribution function $f_X(m)$ versus m for n = 10 and p = q = 1/2. Describe your graph

8. The **geometric** random variable is another example of a random variable defined on Bernoulli experiments. Its value is the trial at which the first "success" occurs. If, for example, the first "success" occurs at the second trial, then the value of the geometric random variable is x=2.

   Now suppose we again flip a fair coin with success again given by a head
   a. Set up a Table with the values of the geometric and binomial random variables for the following outcomes

   HHTT
   THHT
   TTTH

   b. What is a possible outcome if the value of the geometric random variable is 3 and the binomial random variable is 2 after a coin is flipped 5 times

9. Suppose we flip a fair coin until we get a head
   a. Calculate and plot $f_X(1)$, $f_X(2)$ and $f_X(3)$ for the geometric random variable X with $f_X(k)$ equal to the probability of the first success at trial k
   b. Then plot the corresponding cumulative distribution function $F_X(x)$
   c. What is $F_X(4) - F_X(2)$ the probability of

10. Generalizing on the result of Problem (9) we see that if X is a geometric random variable then

$$F_X(k) = \sum_{m=1}^{k} f_X(m) = \text{ P(First success comes by trial k)}$$

= P(At least one success by trial k)

= 1 – P(No successes by trial k)

Now suppose we have a Bernoulli experiment with P(success) = 0.6

a. Find the probability for the first success at trial 4
b. Find the probability for the first success by trial 4

11. Now suppose we flip a fair coin until we get a head
    a. What's the probability $f_X(4)$ that the first head is at the 4th toss
    b. What's the probability that the first head is at the 4th toss if the first three tosses are all tails. Explain your answer. Be careful - this is a conditional probability problem
    c. What's the probability that the second head is at the 7th toss if the first head is at the third toss. How is your answer related to the result in part (a). Explain what's going on

12. A discrete probability distribution function $f_X(x)$ is **uniform** if all the probabilities are equal as follows $f_X(x_1) = f_X(x_2) = \cdots = f_X(x_k)$. **Memorize** this definition
    a. Suppose we flip a coin twice. Come up with a random variable for the outcomes that has a uniform probability distribution. Put your result in a Table
    b. Plot $f_X(x)$ and $F_X(x)$ for your example in part (a)