ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 20
JOINT CONTINUOUS RANDOM VARIABLES - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this Investigation is to calculate parameters like correlation and covariance of continuous random variables that appear in many applications including those involving random processes.

1. We begin with a review of marginal probability densities. Find the marginal probabilities $f_X(x)$ and $f_Y(y)$ for

$$f_{XY}(x,y) = \begin{cases} 4e^{-2(x+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

2. As you would expect we define the **covariance** of two continuous random variables X and Y just like we do for discrete random variables as follows

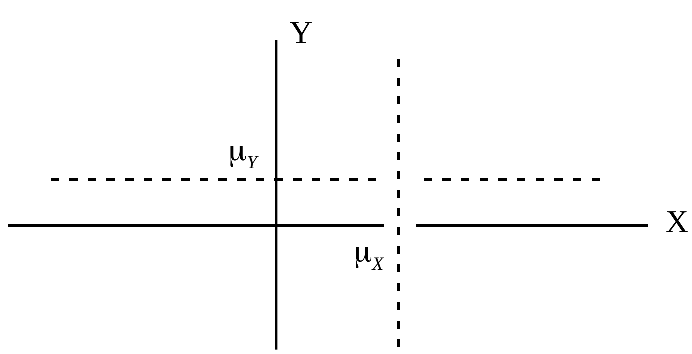
$$\text{Cov}[X, Y] = E[(X - \mu_X)(Y - \mu_Y)]$$

   a. Write out the integral expression for Cov[X, Y]
   b. Make use of your result in part (a) to show that Cov[X, Y] = E[XY] – E[X] E[Y] just like it does in the discrete case

3. And similarly we define the **correlation coefficient** ρ for continuous random variables X and Y just like we do for discrete random variables as follows

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

   Draw a scatter plot for the random variables X and Y on a graph as follows



   a. For $\rho = 0$
   b. $\rho = 1$

4. We now introduce the integral expression for the **correlation** of random variables X and Y as follows

$$R_{XY} = E[XY]$$

   a. How is the correlation R_{XY} different from the correlation coefficient ρ
   b. Under what circumstances is the correlation R_{XY} proportional to ρ
   c. Write out the integral expression for the correlation R_{XY}

5. Random variables X and Y are said to be **uncorrelated** if $R_{XY} = E[X]E[Y]$.
   a. Show that if X and Y are independent, then they're uncorrelated
   b. Show that if X and Y are uncorrelated then Cov [X,Y] = 0

6. Given two random variables X and Y with joint probability density

$$f_{XY}(x,y) = \begin{cases} x(y+1.5) & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   a. Find R_{XY}
   b. Find Cov [X,Y]
   c. Find the correlation coefficient
   c. Make use of your results in parts (a) and (b) to draw a scatter plot of X and Y

7. Given the two continuous random variables X and Y with joint probability density function

$$f_{XY}(x,y) = \begin{cases} 2e^{-x}e^{-y} & 0 \le y \le x \le \infty \\ 0 & \text{otherwise} \end{cases}$$

   Show that $R_{XY} = E[XY] = 1$. Be careful - y goes from 0 to x as x goes from 0 to ∞

8. Find $R_{XY} = E[XY]$ if X and Y have the joint probability density function

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}\left((x-a)^2 + (y-b)^2\right)\right]$$

   Hint - make use of the fact that the integral can expressed as the product of two integrals - one integral that is a function only of x and the other only of y - integrals that should look very familiar.

9. Given the random variables

$$X = \cos\theta \quad \text{and} \quad Y = \sin\theta$$

   with θ uniformly distributed over the interval (0, 2π). Show that $R_{XY} = E[XY] = 0$. Note that

$$E[XY] = \int X(\theta)Y(\theta)d\theta$$

   Hint - make use of a trig identity when calculating the integral

10. Review Problem: From Fourier Series analysis we know that if x(t) is a periodic signal like a pulse train then x(t) can be expressed as a sum of sinusoids as follows

$$x(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(2\pi k f_o t + \theta_k\right)$$

If we then make use of Euler's Relation to express each sinusoid as a sum of complex exponentials as follows

$$c_k \cos\left(2\pi k f_o t + \theta_k\right) = \frac{1}{2} c_k e^{j(2\pi k f_o t + \theta_k)} + \frac{1}{2} c_k e^{-j(2\pi k f_o t + \theta_k)} = X_k e^{j2\pi k f_o t} + X_{-k} e^{-j2\pi k f_o t}$$

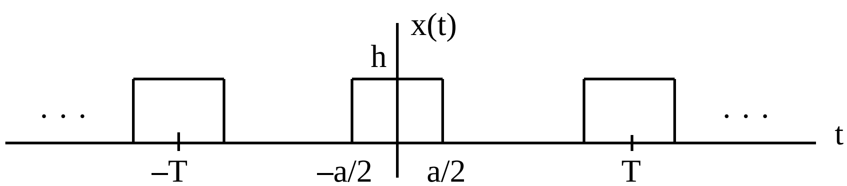
then we can write our Fourier sum of sinusoids as a sum of complex exponentials as follows

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k f_o t}$$

with

$$X_k = \frac{1}{T}\int_T x(t) e^{-j2\pi k f_o t}\, dt$$

If we go through the analysis for a pulse train as follows



we find that

$$X_k = \frac{ha}{T} \operatorname{sinc}(k f_0 a) \qquad \text{where} \qquad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Make use of these results to

a. Sketch $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$
b. Find X_0
c. Find X_1
d. Find X_2