ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 19
JOINT CONTINUOUS RANDOM VARIABLES - PART I

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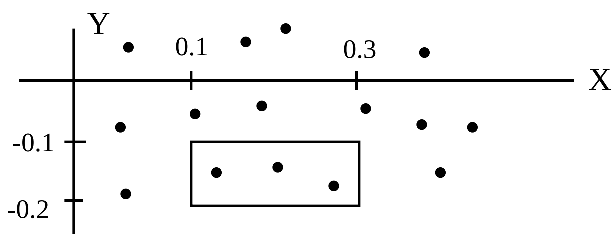
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this invesitgation is to investigate the properties of joint probability density functions $f_{XY}(x,y)$ and joint cumulative distribution functions $F_{XY}(x,y)$ of continuous random variables X and Y. The approach will parallel that of Investigation 12 where we introduced continuous random variables. The results will be analogous to those we obtained for discrete joint probabilities.

1. The objective of this problem is to introduce what we mean by **joint probablity densities**. Suppose we do a random experiment a whole bunch of times with results as follows



What's the joint probability **density** in the rectangular region - the probability that the values of X and Y will be in the rectangular region when we do the random experiment a whole bunch of times divided by the area of the rectangle.

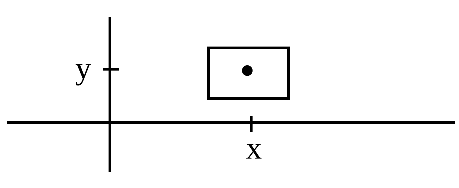
2. Find the joint probability density in the following region around the point (x_o, y_o) if 10 out of every 500 times we do an experiment we find that $x_o - 0.1 \le X \le x_o + 0.1$ and $y_o - 0.05 \le Y \le y_o + 0.05$.

3. The objective of this problem is to make use of the probability density in a region to calculate the joint probability. What, in particular, is the probability that x and y simultaneously satisfy

$$1 \le x \le 1.1 \quad \text{and} \quad 1.5 \le y \le 1.6$$

if the average probability density in this region is 0.2.

4. From Problem (2) we know that the Probability Density in the region around a point (x,y) as follows



is the probability that both X and Y will be in the region when we do the random experiment divided by the area of the region as follows

$$\frac{\text{P[X and Y values are in the Region]}}{\text{Area of Region}}$$

The probability density $f_{XY}(x,y)$ at the point (x,y) is then by definition the limiting value of this ratio as follows

$$f_{XY}(x,y) = \lim_{Area \to 0} \frac{\text{P[X and Y are in the Region]}}{\text{Area of the Region}}$$

a. Write out in words what

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{XY}(x,y)dxdy$$

is the probability of. **Memorize** your result

b. Make use of your result in part (a) to find $P[1 \le X \le 2,\ 0 \le Y \le 1]$ if

$$f_{XY}(x,y) = \begin{cases} 4e^{-2(x+y)} & x \ge 0 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

5. Given the joint probability density $f_{XY}(x,y)$
   a. Explain in words how the following expressions are similar to the marginal probabilities we calculated for discrete random variables

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$

   b. Make use of the results in part (a) to calculate the marginal probabilities $f_X(x)$ and $f_Y(y)$ of

$$f_{XY}(x,y) = \begin{cases} 4e^{-2(x+y)} & x \ge 0 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

6. Given two continuous random variables X and Y
   a. How would you make use of their joint probability density $f_{XY}(x,y)$ to determine if they are independent
   b. Make use of your result in part (a) to determine if X and Y in Problem (5) are independent

7. The objective of this and the rest of the problems of this investigation is to introduce **joint cumulative distribution functions**

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

   and explore the relationship between them and joint densities $f_{XY}(x,y)$. To begin, describe in words what $F_{XY}(x,y)$ is the probability of

8. Make use of your result in Problem (7) to explain in words why $F_{XY}(x,y)$ has the following properties
   a. $F_{XY}(-\infty, -\infty) = 0$
   b. $F_{XY}(x,y)$ is nondecreasing as x and y increase

c. $F_{XY}(\infty, \infty) = 1$

9. Verify that the following joint cumulative distribution function

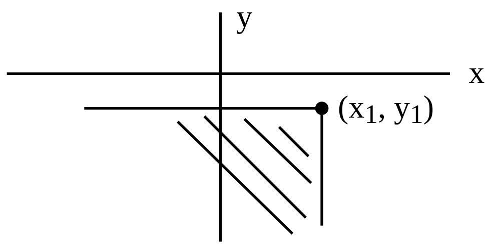
$$F_{XY}(x,y) = \begin{cases} 1 - e^{-2x} - e^{-2y} + e^{-2(x+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

has the properties listed in Problem (8)

10. Make use of your result in Problem (7) to explain in words why
    a. $F_{XY}(x, \infty) = F_X(x)$
    b. $F_{XY}(\infty, y) = F_Y(y)$

11. Find $F_X(x)$ and $F_Y(y)$ for $F_{XY}(x,y)$ of Problem (9)

12. Given the following the xy-plane



a. Describe in words what $F_{XY}(x_1,y_1)$ is the probability of
b. Explain in words why $F_{XY}(x_1,y_1)$ is the probability that (x,y) is in the indicated region in the above graph
c. Find $F_{XY}(1, 2)$ for when

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-2x} - e^{-2y} + e^{-2(x+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

13. Draw a picture of the region in the xy plane corresponding to

$$P(x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2)$$

and then make use of the result in Problem (12) to express this probability as a sum and difference of terms $F_{XY}(x,y)$ - you should end up with a total of four terms

14. Make use of the result

$$P(x_1 < x \leq x_2, y_1 < y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x,y)\,dx\,dy$$

to show that

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x,y)\,dx\,dy$$

Note that we can get $f_{XY}(x,y)$ from $F_{XY}(x,y)$ by calculating the derivative

$$f_{XY}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

15. Given the joint cumulative distribution function $F_{XY}(x,y)$ in Problem (9) as follows

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-2x} - e^{-2y} + e^{-2(x+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

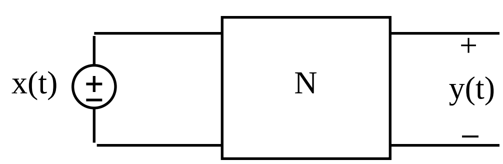
   a. Find $f_{XY}(x,y)$
   b. Make use of your result in part (a) to find $P(0 \leq x \leq 1, 1 \leq y \leq 2)$
   c. Verify that your result in Problem (13) gives the same answer you just got in part (b)

16. Given two random variables X and Y with joint probability density function $f_{XY}(x,y)$
   a. Write out a general expression for E[XY]
   b. Find E[XY] if X and Y have joint probability density function as follows

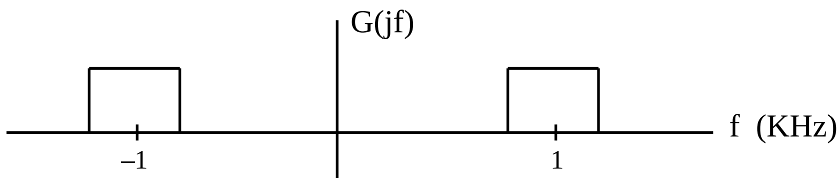
$$f_{XY}(x,y) = \begin{cases} 4e^{-2(x+y)} & x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

17. In preparation for the next Investigation
   a. Find $E[\cos(\theta)]$ if θ is uniform in the interval $-\pi \leq \theta \leq \pi$
   b. Find $E[\cos(\theta)\sin(\theta)]$ if θ is uniform in the interval $-\pi/2 \leq \theta \leq \pi/2$. Hint - make use of appropriate trig identities

18. Review Problem: Sketch the steady state response of the following circuit



with transfer function as follows



when the input x(t) is periodic as follows