ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 18
MONTONIC FUNCTIONS OF RANDOM VARIABLES

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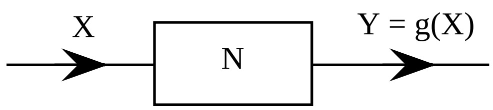
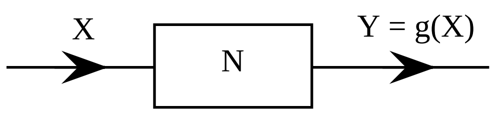
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last couple of investigations we introduced some basic probability density functions and their basic properties. The objective of this investigation is to develop a method for finding the probability density functions $f_Y(y)$ of functions of random variables X as follows



Once we know how to do this, we can then calculate the probability density functions at the outputs of circuits like amplifiers, limiters, rectifiers and so on in terms of the probability density functions $f_X(x)$ of their inputs.

1. The objective of this first problem is to illustrate how to find the the probability distribution of Y = g(X) in a system as follows



in the case where X is discrete. As we know from previous Investigations the probability that Y = y is simply the sum

$$f_Y(y) = \sum f_X(x)$$

where the sum is taken over all x such that y = g(x). Use this result to find $f_Y(y)$ if Y = g(X) is given by

X	**Y = G(x)**
0	1
1	2
2	1
3	3

and $f_X(x)$ is given by

X	**$f_X(x)$**
0	0.1
1	0.2
2	0.4
3	0.3

Put your results in a Table.

2. The objective of this problem is to illustrate how to find the probability density function $f_Y(y)$ of the continuous random variable Y= g(X) when X is a continuous random variable with probability density function $f_X(x)$. The trick is to

   (1) First find $F_Y(y)$ = P[Y ≤ y] in terms of F_X

   (2) And then differentiate to find $f_Y(y) = \dfrac{dF_Y(y)}{dy}$

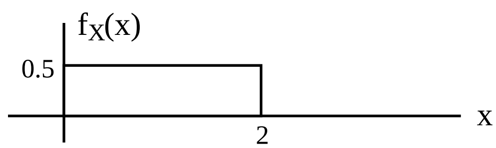
   Suppose in particular that Y = g(X) = 2X. Then

$$F_Y(y) = P[Y \le y] = P[2X \le y] = P\left[X \le \frac{y}{2}\right] = F_X\left(\frac{y}{2}\right)$$

   a. Carry out the differentiation to show that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2} f_X\left(\frac{y}{2}\right)$$

   b. Now make use of your result in part (a) to sketch $f_Y(y)$ when X has a uniform probability density function in the range 0 ≤ X ≤ 2 as follows



3. Generalize on your result in Problem (2) to show that when Y = aX then

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y}{a}\right)$$

4. Now suppose that Y = aX + b
   a. First express $F_Y(y)$ = P[ax + b ≤ y] in terms of F_X
   b. Then make use of your result in part (a) to show that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

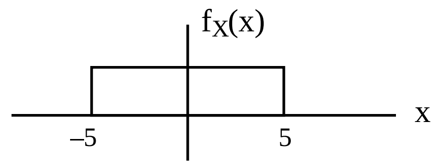
   c. Now find and sketch $f_Y(y)$ if Y = 3X - 2 and X is exponential with $\lambda / T = 2$
   d. Then find and sketch $f_Y(y)$ if Y = 3X - 2 and X is Gaussian with $\mu_X = 0$ and $\sigma_X = 2$.
   e. What is μ_Y and σ_Y in part (d)

5. The objective of this problem is to find the probability density function of the random variable Y at the output of a rectifier with Y = |X|
   a. First express $F_Y(y)$ = P[|x| ≤ y] in terms of F_X. Hint - $P[|x| \le y] = P[-y \le x \le y]$
   b. Then carefully make use of your result in part (a) to show that

$$f_Y(y) = f_X(y) + f_X(-y)$$

   c. Make use of your result in part (b) to find $f_Y(y)$ when $f_X(x)$ is an even function with $f_X(x) = f_X(-x)$

d. Sketch $f_Y(y)$ when $f_X(x)$ has the following probability density. Remember that $y \geq 0$

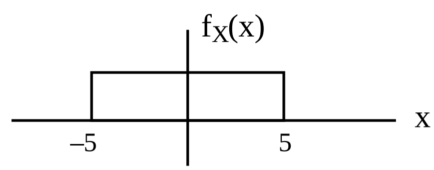


e. Find $f_Y(y)$ at the output of the rectifier if X is Gaussian with $\mu_X = 5$ and $\sigma_X = 2$.

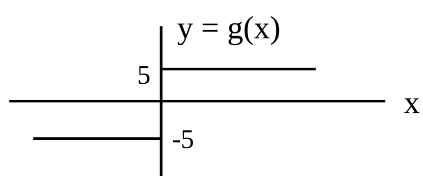
6. The objective of this problem is to find the probability density function of the random variable Y at the output of a squaring circuit with $Y = X^2$
   a. First express $F_Y(y) = P[x^2 \leq y]$ in terms of F_X. Hint - $P\left[x^2 \leq y\right] = P\left[-\sqrt{y} \leq x \leq \sqrt{y}\right]$
   b. Then make use of your result in part (a) to show that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2\sqrt{y}}\left(f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right)$$

   c. Sketch $f_Y(y)$ when $f_X(x)$ has the following probability density. Remember that $y \geq 0$



7. The objective of this problem is to find the probability density function of the random variable Y at the output of a comparator as follows

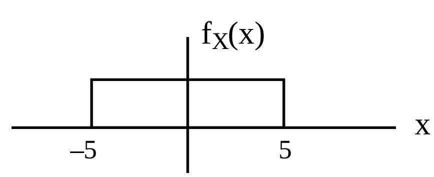


$$Y = \begin{cases} 5 & x > 0 \\ -5 & x \leq 0 \end{cases}$$

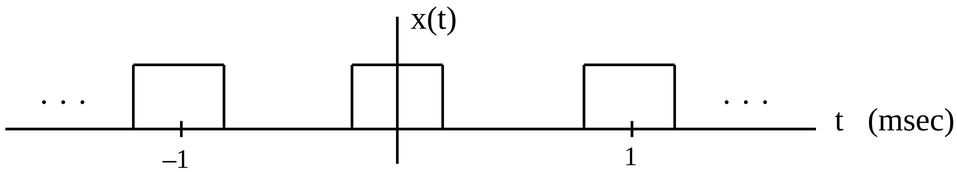
   a. First make a sketch of $F_Y(y)$. Hint - make use of the fact that

$$F_Y(y) = \begin{cases} 0 & y < -5 \\ F_X(0) & y = -5 \\ F_X(0) & -5 < y < 5 \\ 1 & y \geq 5 \end{cases}$$

   b. Then make use of your result in part (a) to find $f_Y(y)$
   c. Sketch $f_Y(y)$ when $f_X(x)$ has the following probability density



8. Review Problem: Given the following pulse train x(t)



a. Why do we say x(t) is periodic
b. What is the fundamental frequency f_o of x(t) in Hz
c. What is the frequency of the third harmonic in Hz
d. What does Fourier tell us about periodic signals like our pulse train