ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 17
THE CENTRAL LIMIT THEOREM

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

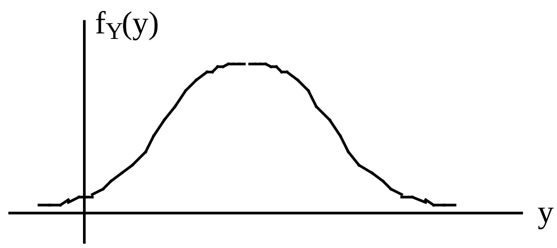
From the last investigation, we know that the binomial distribution as follows

$$f_X(k) = \text{sum of successes in n independent Bernoulli trials}$$

can be approximated by a Gaussian distribution as long as n is "large enough" - as long as npq > 10. What is even more significant is that it doesn't matter what the random variables in the sum are equal to - whether they're geometric, uniform, Poisson, exponential or whatever. As long they're independent and identically distributed (**iid**) and n is "large enough" then Y equal to the following sum

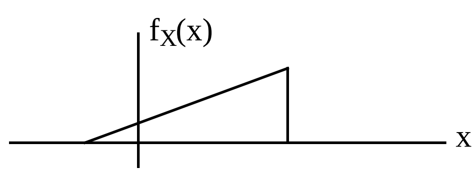
$$Y = X_1 + X_2 + \ldots + X_n$$

will be approximately Gaussian as follows



with mean $\mu_Y = n\mu_X$ and variance $\sigma_Y^2 = n\sigma_X^2$ where μ_X is the mean of the random variables X_k and σ_X^2 is their variance. Note that the X_i's can also be n independent observations of the same random variable or the results from performing the same experiment n independent times. The key factor is that a "whole bunch" of identical independent random variables contribute to the value of Y. We call this result the **Central Limit Theorem**.

1. Suppose the random variables $X_1, X_2, \ldots, X_n$ all have the following independent identical density functions

   Then sketch the probability density function of $Y = X_1 + X_2 + \ldots + X_n$

2. Suppose $X_1, X_2, \ldots, X_5$ are independent, identically distributed Gaussian random variables with $\mu_X = 5$ and $\sigma_X = 2$.
   a. Draw a graph of the probability density functions of the random variables X
   b. Find μ_y and σ_y for $Y = X_1 + X_2 + X_3 + X_4 + X_5$

c. Draw a graph of the probability density function for Y
d. Find $P[22 < Y < 27]$

3. Suppose we have a digital voltmeter set to a scale such that every time we take a reading it rounds off the result to the nearest tenth of a volt. And that the random variable X_i equal to the error at the i'th reading is uniformly distributed over the range (-0.05, 0.05).
   a. Draw a graph of the probability density functions of X_i
   b. Draw a graph of the probability density of $Y = X_1 + X_2 + \cdots + X_{750}$ equal to a sum of 750 such readings
   c. What will be the probability that the magnitude of the error of our sum of 750 readings will be greater than one volt. Indicate the area of $f_Y(y)$ of interest. Hint - make use of the result in Investigation 13 for the variances of uniform distributions. This result is of particular interest in digital signal processing. Note that the errors will tend to cancel out when we sum them

4. Suppose orders from a restaurant are iid random variables with μ = \$10 and standard deviation σ = \$2. Then after drawing probability density functions as usual
   a. Estimate the probability that the first 100 customers will spend at least \$1025. Indicate the corresponding area on a graph of $f_Y(y)$.
   b. What's the probability that the first 100 customers spend between \$975 and \$1025. Indicate the corresponding area on a graph of $f_Y(y)$.
   c. How many customers will it take before we can be 90% sure that the customers have spent at least \$1200. Indicate the corresponding area on a graph of $f_Y(y)$.

5. A fair coin is tossed 1000 times. Estimate the probability that the number of heads is between 450 and 550. Be sure to draw graphs as above

6. The number of messages arriving at a computer is a Poisson random variable with a mean of 10 messages/second. Use the central limit theorem to estimate the probability that more than 610 messages arrive in one minute. Be sure to draw graphs as above

7. Review Problem: Make use of Euler's Relation as follows

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

   to

   a. Express $3e^{j(1000t+1.2)}$ as a cosine plus a sine
   b. Express $x(t) = \cos(2\pi 5000t + 1.2)$ as the real part of a complex exponential
   c. Express $3e^{j1.2}e^{j1000t}$ as a cosine plus a sine
   d. Find $\mathrm{Re}\left[3e^{j1.2}e^{j1000t}\right]$
   e. Show that $A\cos(2\pi ft + \theta) = \frac{1}{2}e^{j(2\pi ft+\theta)} + \frac{1}{2}e^{-j(2\pi ft+\theta)}$

8. Review Problem:
   a. What is the chain rule for differentiating a function of a function
   b. Illustrate with an example