ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 15
THE GAUSSIAN DISTRIBUTION - PART I

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

As we know from Investigation 9, the Poisson distribution is the limiting case of the binomial distribution when there are only a few successes in a very large number of Bernoulli trials. The objective of this Investigation is to make use of the binomial distribution to introduce the normal or Gaussian probability distribution. The normal distrtibution is particularly important because we see it in so many applications - everything from the distribution of grades to the distribution of circuit element values to the distribution of random noise.

1. Given a binomial distribution with p = 0.5
   a. Sketch the discrete probability distribution $f_X(k)$ as a function of k for n = 10
   b. Describe what your graph looks like

2. Going through the math we can show that as n increases the values of the binomial probability distribution as follows

$$f_X(k) = \binom{n}{k} p^k q^{n-k}$$

   get closer and closer to lying on the **Gaussian** or **normal** probability density function as follows

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

   with $\mu = np$ and $\sigma = \sqrt{npq}$. The Gaussian density function is referred to as the **normal distribution** because it's so common - because there are so many situations where the outcomes are the sum of a whole bunch of independent random trials like the flipping of a coin or rolling of a die a whole bunch of times. The objective of this problem is to see what the Gaussian density function looks like
   a. Find $f_X(\mu)$
   b. Show that $f_X(x)$ is symmetric about x = μ
   c. Show that $f_X(x)$ goes to zero as x goes to plus and minus infinity
   d. Explain how we can tell that $f_X(x)$ is maximum at x = μ
   e. Explain how we can tell that $f_X(x)$ will be narrow if σ is small
   f. Make use of your results in parts (a)–(d) to sketch the continuous probability density function $f_X(x)$ as a function of x for μ = 100 and σ = 20
   g. Why is the Gaussian density function referred to as a bell shaped curve
   h. As you can undoubtedly tell from the choice of symbols, μ is the mean and σ^2 the variance of our Gaussian distribution. Note that σ is referred to as the **standard deviation** of the distribution. Sketch $f_X(x)$ for μ = 0 and a small value of σ and then for a large value of σ. Describe the difference between the two curves

i. Make use of Mathcad to obtain graphs of $f_X(x)$ for small and large values of σ on the graph. Describe the difference between the two curves

3. Given a Gaussian distribution with μ = 0 and σ = 10
   a. Sketch the Gaussian density function $f_X(x)$
   b. Shade in the region corresponding to the probability P(x ≥ 10). How does this compare to the region corresponding to the probability P(x ≤ –10)
   c. How would you find P(x ≥ 10) if all you had was $F_X(10)$. Indicate what's going on with a picture of $f_X(x)$
   d. How would you find P(–10 ≤ x ≤ 10) if all you had was $F_X(10)$. Draw a picture to illustrate

4. Write out the expression for the integral for the cumulative probability function $F_X(x)$ of the Gaussian distribution. **Don't** try to solve it.

5. The difficulty with the integral for $F_X(x)$ in Problem (4) is that it has no closed form solution and so must be evaluated with numerical methods. The text contains a Table for the **normalized** Gaussian distribution - Gaussian distributions with μ = 0 and σ = 1. Note that you can also use your calculator to calculate the integral for $F_X(x)$

   Show that with the change of variable

$$z = \frac{x - \mu}{\sigma}$$

   the integral

$$F_X(b) - F_X(a) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

   becomes

$$F_X(b) - F_X(a) = F_Z\left(\frac{b-\mu}{\sigma}\right) - F_Z\left(\frac{a-\mu}{\sigma}\right) = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}\, dz = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} f_z(z)\, dz$$

   where

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}\, e^{-z^2/2}$$

   is the probability density of a normalized Gaussian distribution with μ = 0 and σ = 1

6. The objective of this problem is to make use of the normalized Gaussian distribution for finding P(5 ≤ x ≤ 8) for a Gaussian distribution with μ = 5 and σ = 2
   a. Draw the probability density function $f_X(x)$. Shade in the region corresponding to P(5 ≤ x ≤ 8)
   b. Draw the normalized density function $f_Z(z)$ with the corresponding region shaded in
   c. Make use of a Table of the normalized Gaussian distribution to find the probability

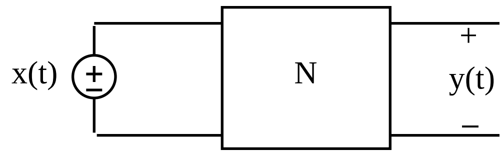
7. Suppose that the errors in the readings of a given voltmeter

$$v_{error} = v_{meas} - v_{actual}$$

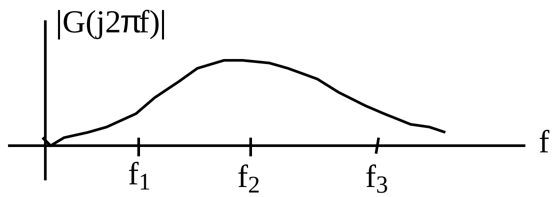
   have a normal distribution with mean μ = 0 and standard deviation σ = 0.1 volt

a. What's the probability that $v_{error} > 0.15$ (that a measurement will be greater than the actual voltage by at least 0.15 volts). Indicate this region on a graph of the probability density function
b. What's the probability that $v_{error} < -0.15$ (that a measurement will be less than the actual voltage by at least 0.15 volts). Indicate this region on a graph of the probability density function
c. What's the probability that the magnitude of the errors will be less than 0.15 volts. Draw the probability density function and indicate the region of interest

8. Review Problem: Given the following 2nd order circuit



with frequency response as follows



Sketch $x(t) = \cos(2\pi ft)$ and the steady state response y(t) at the frequencies

a. $f = f_1$
b. $f = f_2$
c. $f = f_3$