ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 14
THE EXPONENTIAL DISTRIBUTION

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this and the next several Investigations is to introduce some important continuous random variables. The objective of this Investigation, in particular, is to introduce the continuous exponential probability density function $f_X(x)$ for finding the probability that the first (or next) success in a Poisson experiment comes by time x.

1. We begin with a review the geometric distribution for Bernoulli experiments. Suppose we're flipping a coin with P(H) = 0.6. And by success we mean a head.
   a. What's the probability $f_X(1)$ that the first success occurs on the first trial
   b. What's the probability $f_X(2)$ that the first success occurs on the second trial
   c. What's the probability $F_X(2)$ that the first success occurs by the second trial

2. From Problem (1) we know that

$$\begin{aligned} F_X(2) &= \text{P(First success comes by the second trial)} \\ &= \text{P(At least one success by the second trial)} \\ &= 1 - \text{P(No successes in first two trials)} \end{aligned}$$

   Make use of this result to find $F_X(k)$ = P(First success comes by the k'th trial) when we're flipping a coin with P(H) = p and by success we mean a head

3. We now do an analogous review for the Poisson distribution. Given a Poisson distribution with an average of 5 calls every 3 minutes
   a. Find the average number of calls in two minutes
   b. Find the probability of one success in the first two minutes
   c. Find the probability that the first success occurs in the first two minutes

4. Generalizing on the results of Problem (3) we have that if $f_X(k)$ is a Poisson probability distribution with an average of λ successes every T minutes then the average number of calls in x minutes is $x(\lambda/T)$ and so the probability of k successes in x minutes is given by

$$f_X(k) = \frac{\left(x\frac{\lambda}{T}\right)^k}{k!}\, e^{-\left(x\frac{\lambda}{T}\right)}$$

   Make use of this relation to
   a. Find the probability of one success in x minutes
   b. Find the probability that the first success comes in x minutes

5. The objective of this problem is to find an expression for the probability density function $f_X(x)$ for finding the probability that the first success in a Poisson experiment comes by time x. Our plan is to first find $F_X(x)$ and then make use of it to find $f_X(x)$. Now let us suppose, to keep things simple, that the "last success" was at time x = 0. Then the cumulative probability $F_X(x)$

as follows

$$F_X(x) = \int_0^x f_X(x)dx$$

is equal to the probability that the next success comes by time x.

a. Make use of the fact that $F_X(x)$ is the probability of at least one success by time x to justify why

$$F_X(x) = 1 - \text{(Probability of no successes in the interval 0 to x)}$$

b. Make use of your result in part (a) to obtain an expression for $F_X(x)$
c. Make use of your result in part (b) to show that

$$f_X(x) = \frac{\lambda}{T} e^{-\lambda(x/T)} = \frac{\lambda}{T} e^{-x(\lambda/T)}$$

d. Sketch $f_X(x)$
e. Why do we refer to $f_X(x)$ for the time between successes in a Poisson distribution as the **exponential probability density**

6. Explain why the Poisson distribution is discrete and the exponential distribution continuous. Illustrate with a graph of $f_X(k)$ for the Poisson and a graph of $f_X(x)$ for the exponential

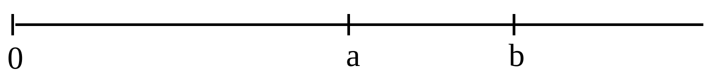
7. Find the probability that the first phone call in a Poisson distribution occurs in the first two minutes if an average of 5 phone calls are arriving every 3 minutes

8. Explain in words what the following expression is the probability of

$$F_X(b) - F_X(a) = \int_a^b f_X(x)dx$$

9. Find the probability that the first phone call in a Poisson distribution occurs in the interval $1 \le x \le 3$ if an average of 3 phone calls are arriving every two minutes

10. This problem generalizes on Problem (9). Given a Poisson distribution with parameters λ and T and times a and b as follows



find

a. P(X ≤ a) = Probability of first success by time a
b. P(X ≤ b) = Probability of first success by time b
c. P(a ≤ X ≤ b) = Probability of first success between times a and b
d. P(X ≥ b) = Probability of first success after time b

11. Suppose phone calls are arriving at an average of 10 calls/minute on a particular line
    a. What's the probability of at least one phone call over any given 10 second period. Note that this is really a conditional probability problem
    b. What's the probability that the first phone call occurs between time t = 2 seconds and t = 4 seconds
    c. How long do we have to wait before the probability of at least one phone call having arrived is 0.9

12. Suppose a communication channel has an average of one failure per year. What's the

probability that it will have at least one failure within the next 100 days. Be sure to keep your units consistent. Use either days or years as your basic unit.

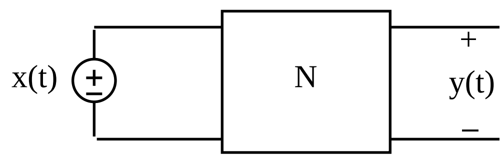
13. Suppose the flaws in a magnetic tape have a Poisson distribution with an average of λ = 0.01 flaws/inch. What's the probability of a flaw within 50 inches after a given flaw

14. Suppose phone calls are arriving at an average of 10 calls/minute. What's the average time between calls

15. Generalize on your result in Problem (13) to show that the average time between successes in a Poisson experiment is T/λ as follows

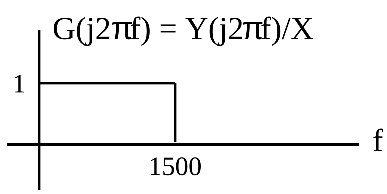
$$E[X] = \int_0^x x\,\frac{\lambda}{T}\,e^{-x(\lambda/T)}\,dx = \frac{T}{\lambda}$$

16. Review Problem: Given the following circuit

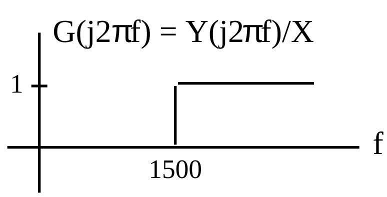


with x(t) = 2 cos (2π1000t) + cos (2π2000t)

a. Sketch x(t)
b. Find and sketch the steady state y(t) if N has the following transfer function



c. Find and sketch the steady state y(t) if N has the following transfer function



16. Math Review: Given the following integral

$$A = \int_2^7 5x\,dx$$

a. Calculate A
b. Write out the integral with the change of variable y = 5x
c. Calculate your integral in part (b)
d. Verify that your results in parts (a) and (c) are the same

17. Math Review: Sketch $y = e^{-x^2}$ for $-2 \le x \le 2$