ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 13
BASIC PROPERTIES OF CONTINUOUS RANDOM VARIABLES

WINTER 2004 **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of the last investigation was to introduce probability density functions $f_X(x)$ and cumulative probability functions $F_X(x)$ for continuous random variables X. Our main results were

$$F_X(b) - F_X(a) = P(a < x \le b) = \int_a^b f_X(x)dx \quad \text{and} \quad F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)dx$$

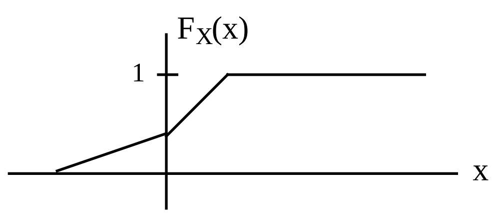
The main objectives of this investigation are to first introduce equations for calculating expectations and variances of continuous random variables and then show that discrete random variables are really special cases of continuous random variables.

1. We begin with some review problems. Given the following cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-2x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

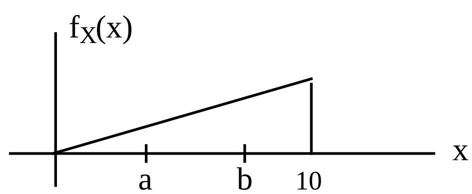
   a. Sketch $F_X(x)$
   b. Find the probability density function $f_X(x)$
   c. Sketch your $f_X(x)$ in part (b)

2. Given the following cumulative distribution function



   Sketch the probability density function $f_X(x)$

3. Given the following probability density



   a. Sketch $F_X(x)$
   b. Find an expression for the probability $P(a \le x \le b)$ as a function of a and b

4. The objective of this problem is to introduce formulas for calculating expectations and variances

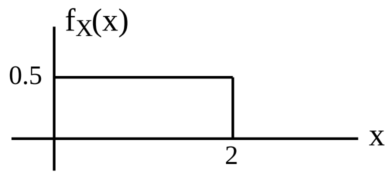
of continuous random variables.

a. Describe in words the similarities between the following integrals for calculating expectations and variances of continuous random variables

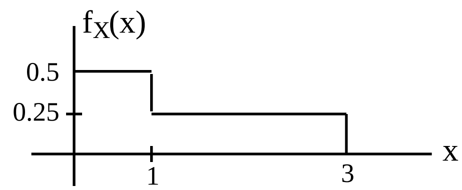
$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \qquad Var[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

and the summations we use to calculate the expectations and variances of discrete random variables. Note that you will often see the variance expressed as $\sigma^2 = Var[x]$ where σ is referred to as the **standard deviation**

b. Find E[X] and Var[X] for a random variable X with uniform probability density function as follows



c. Find E[X] and Var[X] for a random variable X with probability density function as follows



5. Show that continuous random variables satisfy $Var[X] = E\left[X^2\right] - E^2[X]$ just like discrete random variables do

6. Make use of the result of Problem (5) to find Var[X] for a continuous random variable with probability density function $f_X(x)$ that is uniform (constant) over the range x = 0 to x = 10

7. Show that if X is a continuous random variable with uniform probability over the range x = a to x = b then

   a. $E[X] = \dfrac{a+b}{2}$

   b. $Var[X] = \dfrac{(b-a)^2}{12}$

8. Describe in words how the following integral for the expectation of the continuous random variable g(x)

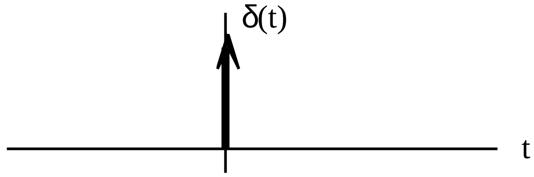
$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

is similar to the summation used to calculate the expectation of g(x) for discrtete variables

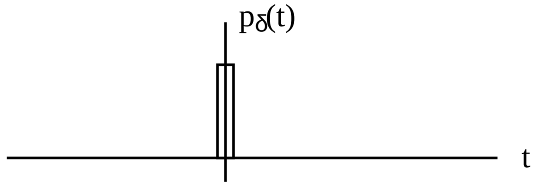
9. Make use of the result in Problem (8) to show that continuous random variables satisfy E[kX] = kE[X] for constants k just like discrete random variables do

10. The objective of this and the rest of the problems in this Investigation is to introduce

continuous random variables with impulses. We begin with a review of the impulse function $\delta(x)$. As we know from networks impulse functions $\delta(x)$ as follows



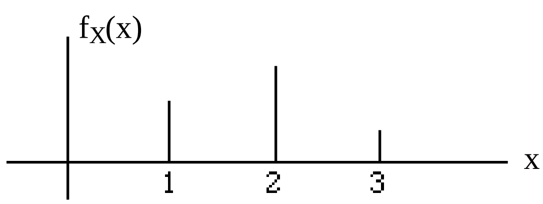
are - for purposes of calculating with them - equivalent to very, very narrow pulses $p_\delta(t)$ of width ε, height $1/\varepsilon$ and area one as follows



Make use of this result to

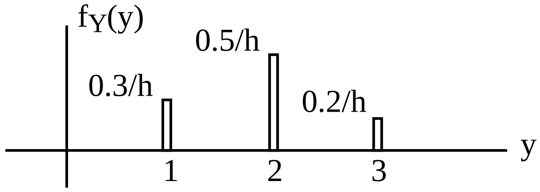
a. Sketch $x(t) = \delta(t - 1)$
b. Sketch $x(t) = \delta(t + 1)$
c. Sketch $x(t) = 2\delta(t - 2)$
d. Sketch $x(t) = 2\delta(t + 1) + 3\delta(t - 2)$
e. Find $a = \int_0^{10} 3\delta(t-4)dt$

11. The objective of this problem is to see how impulses can be used to generate continuous random variables with probability density functions that are equivalent to the probability distribution functions of discrete random variables. Suppose, for example, that X is a discrete random variable with probability distribution $f_X(x)$ as follows



with probabilities $f_X(1) = 0.3$, $f_X(2) = 0.5$, $f_X(3) = 0.2$

a. Let us begin by constructing a "corresponding" continuous random variable Y with probability density function $f_Y(y)$ as follows



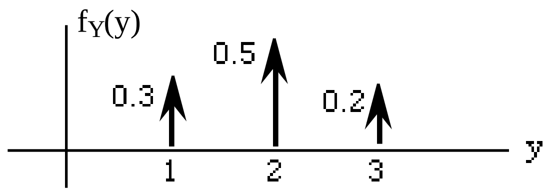
that is equal to zero everywhere except near $y = 1, 2$ and 3 where there are very narrow pulses of width h and heights as shown. Find each of the following probabilities

(i) $P(1 - h/2 \le y \le 1 + h/2)$
(ii) $P(2 - h/2 \le y \le 2 + h/2)$
(iii) $P(3 - h/2 \le y \le 3 + h/2)$

b. Explain in words why the continuous probability density function in part (a) is a good approximation to the discrete distribution function $f_X(x)$ when h is small

c. Taking the limit as $h \to 0$ in part (a), $f_Y(y)$ becomes the sum of impulses

$f_Y(y) = 0.3\,\delta(y - 1) + 0.5\,\delta(y - 2) + 0.2\,\delta(y - 3)$ as follows



Explain why this probability density function is equivalent to the discrete $f_X(x)$

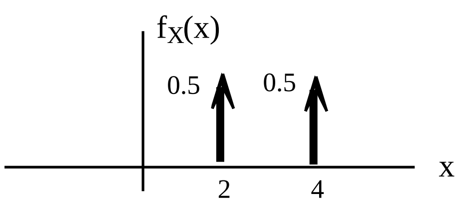
d. Now make use of the result in part (c) to graph $F_Y(y)$ and verify that it's the same as $F_X(x)$ for the discrete case

e. Write out an equation for and then plot the continuous probability density function corresponding to the discrete random variable X with probabilities

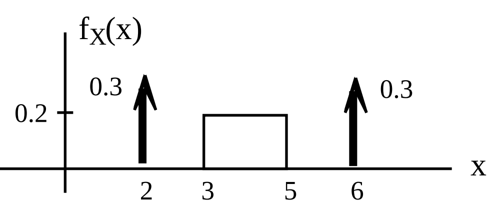
$f_X(1) = 0.3 \qquad f_X(2) = 0.4 \qquad f_X(3) = 0.1 \qquad f_X(4) = 0.2$

f. Sketch $F_X(x)$ for $f_X(x) = 0.2\delta(x) + 0.3\delta(x-2) + 0.5\delta(x-5)$

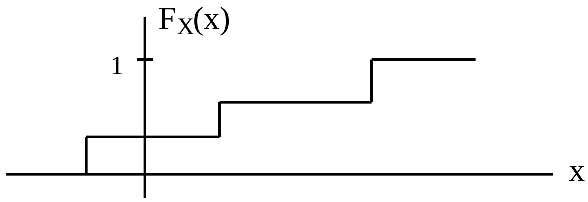
g. Sketch $F_X(x)$ for $f_X(x)$ given by



h. Sketch $F_X(x)$ for $f_X(x)$ given by



i. Find and sketch $f_X(x)$ with $F_X(x)$ as follows



12. The objective of this problem is to show how to evaluate the following integral

$$\int_{-\infty}^{\infty} 0.2\, x\,\delta(x-3)\,dx$$

so we can calculate expectations and variances of probability density functions containing impulses.

   a. First draw a graph of $f(x) = 0.2x$
   b. Now draw a graph of $\delta(x - 3)$ approximated by a very narrow pulse of area one that is centered at $x = 3$
   c. Now make use of your results in parts (a) and (b) to draw a graph for approximating the product $0.2\, x\, \delta(x - 3)$
   d. Now make use of your graph in part (c) to obtain the value for the integral

$$\int_{-\infty}^{\infty} 0.2\, x\,\delta(x-3)\,dx$$

   Explain in words how you got your result

13. Generalizing on the result of Problem (11) it can be shown that

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_o)dx = f(x_o)$$

Make use of this result to calculate

$$a = \int_{-\infty}^{\infty} (x-1.5)^2\,\delta(x-2)\,dx$$

14. Given the following probability density functions

$$f_X(x) = 0.3\,\delta(x-1) + 0.5\,\delta(x-2) + 0.2\,\delta(x-3)$$

   a. Use of your results in Problems (5) and (13) to find the expectation and variance of X
   b. Now verify that you get the same result when you calculate the expectation and variance of the corresponding discrete random variable

15. Review Problem: Sketch the magnitudes of each of the following transfer functions as a function of frequency f

   a. $G(j2\pi f) = \dfrac{Y(j2\pi f)}{X} = \dfrac{1500}{1500 + j2\pi f}$

   b. $G(j2\pi f) = \dfrac{Y(j2\pi f)}{X} = \dfrac{j2\pi f}{1500 + j2\pi f}$