ECE 315 - CONTINUOUS RANDOM VARIABLES - INVEST 12
FROM DISCRETE TO CONTINUOUS RANDOM VARIABLES

WINTER 2004 **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

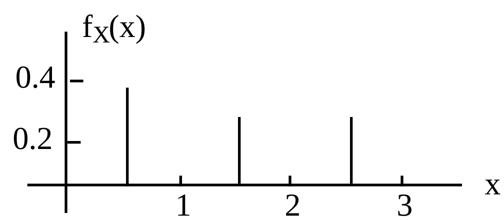
Up to now we've only been working with discrete random variables. The objective of this investigation is to introduce continuous random variables and their probability density functions. The basic difference between discrete and continuous random variables is that we obtain the values x of discrete random variables by counting things like the number of heads when we flip a coin. And we obtain the values X of continuous random variables like heights and weights.

1. Why is the height of a student chosen at random a continuous rather than a discrete random variable

2. Find your own example of a continuous random variable. Explain why it's continuous rather than discrete

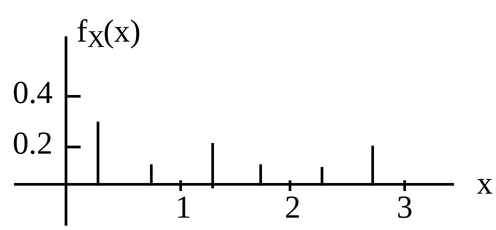
3. The objective of this problem is to introduce *staircase probability density functions* with an example. Suppose that when we randomly choose rocks from a big pile of rocks we obtain the following probability distribution for their weights

Wt. (Kg)	**X**	**$f_X(x)$**
0-1	0.5	0.4
1-2	1.5	0.3
2-3	2.5	0.3

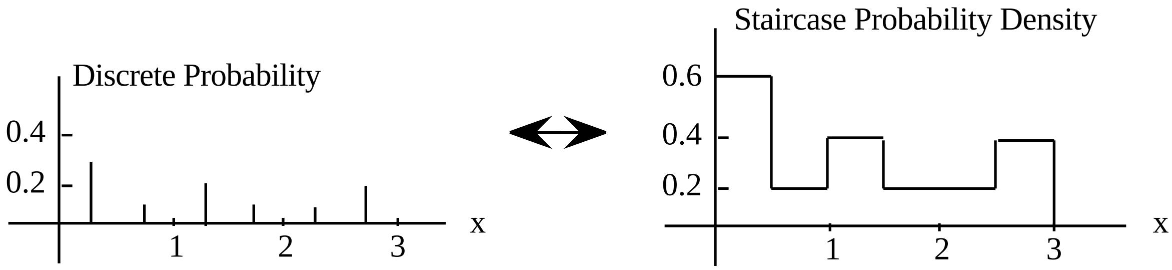


where the values of the discrete random variable X are the midpoints of the intervals. Now suppose we increase the resolution from 1Kg to 0.5Kg and obtain the more detailed probability distribution function given by

Wt.(Kg)	**X**	**$f_X(x)$**
0.0-0.5	0.25	0.3
0.5-1.0	0.75	0.1
1.0-1.5	1.25	0.2
1.5-2.0	1.75	0.1
2.0-2.5	2.25	0.1
2.5-3.0	2.75	0.2



where again the values of X are the midpoints of the intervals. As we continue this process we'll get higher and higher resolution but our graphs are going to get harder and harder to read as the probabilities get smaller and smaller. The way we get around this problem is to replace the graphs of discrete probabilities by graphs of *staircase probability density functions* as follows

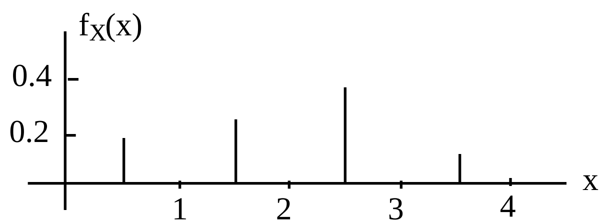


where the value of the staircase probability density in an interval $a \le x \le b$ is equal to the probability that a rock has a weight in that interval divided by the size of the interval as follows

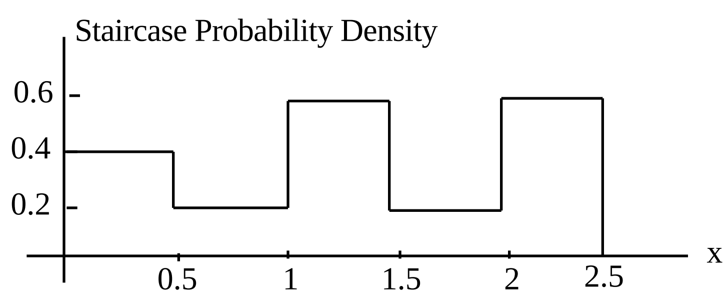
$$\frac{P(a < x \le b)}{b - a} = \frac{F_X(b) - F_X(a)}{b - a}$$

a. Find and plot the staircase probability density function with the following discrete probability distribution

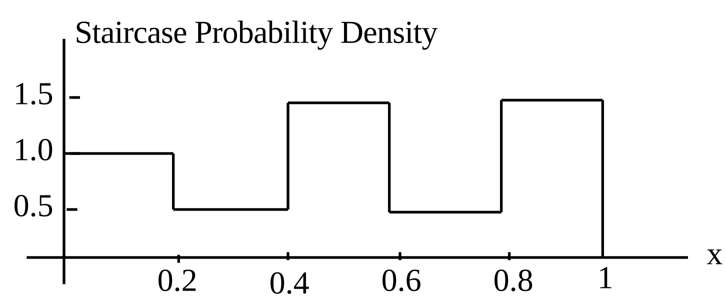
X	$\mathbf{f_X(x)}$
0.5	0.2
1.5	0.3
2.5	0.4
3.5	0.1



b. Find and plot the discrete probability distribution with the following staircase probability density function

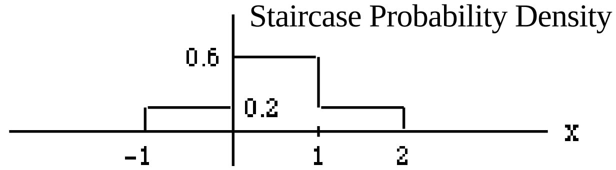


c. Given the following the following staircase probability density function



(i) Find $P(0.2 \le x \le 0.4)$
(ii) Find $P(0.2 \le x \le 0.6)$

4. Given the following staircase probability density function



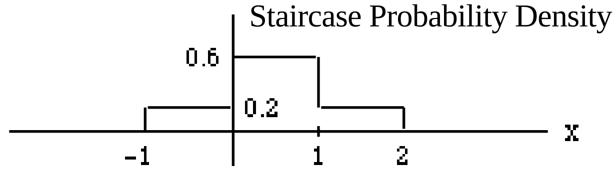
a. Verify that the area under the curve is equal to one
b. Justify the fact that the area under a staircase probability density function is always equal to one. Hint - what is the area in any given interval equal to

5. Generalizing on the previous problems we have that

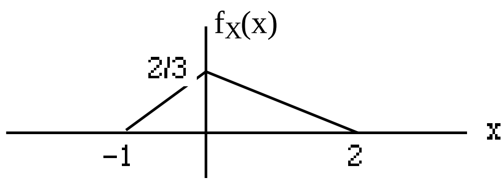
$$\text{Probability density in an interval} = \frac{\text{Probability of an outcome in the interval}}{\text{Length of interval}}$$

Now suppose that 10 out of every 800 times we do a given random experiment the value of the continuous random variable X is in the interval $1.1 \le X \le 1.15$. What is the value of the staircase probability density in this interval.

6. If we repeatedly increase the resolution of a given staircase probability density function like this one



we'll eventually end up in the limit with a "nice" continuous function $f_X(x)$ something like the following



It's this function $f_X(x)$ that we define to be the **probability density function** of the continuous random variable X. **Memorize** this definition
a. Verify that the area under the probability density function above is equal to one
b. Find $P(-1 \le X \le 1)$
c. Find $P(-0.5 \le X \le 1)$
d. Generalizing on the results of parts (c) and (d) we have that

$$P(a < x \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$$

Memorize this result - it's fundamental. Then use it to find

$$P(-1 \le x \le a) \qquad \text{for } a \ge 0$$

e. Make use of your result in part (e) to find $P(X \le 0.5)$

7. Generalizing on the results of Problem (6e) we have that the **cumulative distribution function (cdf)** $F_X(x)$ of a continuous random variable X is given by

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)dx$$

Memorize this expression

a. Now obtain an expression for $f_X(x)$ in terms of $F_X(x)$. **Memorize** your result

b. Explain in words why

$$F_X(-\infty) = 0 \qquad \text{and} \qquad F_X(\infty) = 1$$

c. Sketch $F_X(x)$ for the probability density function of Problem (6). Make sure it's consistent with your results of part (b)

8. Suppose X is a continuous random variable with probability density function $f_X(x)$ that is uniform (constant) over the range $x = 0$ to $x = 10$

a. Draw a graph of $f_X(x)$. Be sure to specify the magnitude of $f_X(x)$

b. Find $P(2 \le x \le 5)$. Describe in words how you got your answer

c. Draw a graph of $F_X(x)$. Explain in words how you got your result

9. Find and sketch $f_X(x)$ for

$$F_X(x) = \begin{cases} 1 - e^{-x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

10. Given the following probability density function

$$f_X(x) = \begin{cases} 0.01e^{-0.01x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch $f_X(x)$

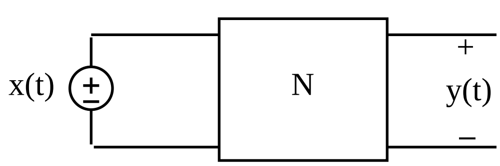
b. Verify that $\int_{-\infty}^{\infty} f(x)dx = 1$

c. Find $P[0 \le x \le 100]$

d. Find and sketch $F_X(x)$

11. Can a continuous probability density function $f_X(x)$ be greater than one in a given interval. Explain why or why not. Illustrate with an example

12. Review Problem: Find the sinusoidal steady state response of the following circuit



with $x(t) = 5\cos(2\pi 1000t + 1.2)$ if the transfer function is

a. $|G(j2\pi f)| = \dfrac{Y(j2\pi f)}{X} = \dfrac{1500}{1500 + j2\pi f}$ b. $|G(j2\pi f)| = \dfrac{Y(j2\pi f)}{X} = \dfrac{j2\pi f}{1500 + j2\pi f}$