ECE 315 - DISCRETE RANDOM VARIABLES - INVEST 10
JOINT PROBABILITY DISTRIBUTIONS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

One of the first things we did in Investigation 1 was to calculate the probability of the intersection of two events A and B as follows P(A ∩ B) - the probability that the result of a random experiment is an outcome in both A and B. The objective of this Investigation is to generalize on these results with the calculation of joint probabilities like $f_{XY}(1,2)$ - the probability that the result of a random experiment is an outcome for which both X=1 and Y=2.

1. The objective of this first problem is to illustrate what we mean by joint probability with an example using the following experimental data from a random experiment conducted a "whole bunch" of times

X	Y
1	2
2	1
2	2
2	1
1	1
1	2
2	1
2	1

   Make use of this data to calculate each of the following **joint probabilities** $f_{XY}(x,y)$ of the random variables X and Y

   a. $f_{XY}(1,1)$ = P(X=1 and Y=1)
   b. $f_{XY}(1,2)$ = P(X=1 and Y=2)
   c. $f_{XY}(2,1)$ = P(X=2 and Y=1)
   d. $f_{XY}(2,2)$ = P(X=2 and Y=2)

2. Suppose we flip two fair coins to generate the two random variables X and Y as follows

X		Y	
0	First coin is a head	0	Second coin is a head
1	First coin is a tail	1	Second coin is a tail

   a. Write out in words what each of the following joint probabilities is the probability of

      $f_{XY}(0,0)$ = P(X=0 and Y=0)
      $f_{XY}(0,1)$ = P(X=0 and Y=1)
      $f_{XY}(1,0)$ = P(X=1 and Y=0)
      $f_{XY}(1,1)$ = P(X=1 and Y=1)

   b. Flip two coins at least 50 times and then make use the results to generate the joint probabilities of part (a) like you did from the data in Problem (1)

c. Put your probabilities from part (b) in a Table of the following form

X \ Y	0	1
0		
1		

d. What information does $f_{XY}(0,1)$ give us that we don't get from just $f_X(0)$ and $f_Y(1)$

e. How can $f_X(0)$ be found from $f_{XY}(0,0)$ and $f_{XY}(0,1)$. Explain

3. Suppose X and Y are random variables such that X can take on the values 1, 2 or 3 and Y can take on the values 1 or 2. And that their joint probabilities are as follows

$f_{XY}(1,1) = P(X=1 \text{ and } Y=1) = 0.1$
$f_{XY}(1,2) = P(X=1 \text{ and } Y=2) = 0.2$
$f_{XY}(2,1) = P(X=2 \text{ and } Y=1) = 0.2$
$f_{XY}(2,2) = P(X=2 \text{ and } Y=2) = 0.1$
$f_{XY}(3,1) = P(X=3 \text{ and } Y=1) = 0.3$
$f_{XY}(3,2) = P(X=3 \text{ and } Y=2) = 0.1$

a. Put the data in a 3x2 Table with entries $f_{XY}(x,y) = P(X=x, Y=y)$

b. Find $f_X(1)$, $f_X(2)$ and $f_X(3)$. Explain and justify how you got your results

c. Find $f_Y(1)$ and $f_Y(2)$. Explain and justify how you got your results

d. Generalizing on the results of part (b) and (c) we have

$$f_X(x) = \sum_y f_{XY}(x,y) \qquad \text{and} \qquad f_Y(y) = \sum_x f_{XY}(x,y)$$

with $f_X(x)$ and $f_Y(y)$ equal to the **marginal** probability distribution functions of $f_{XY}(x,y)$. **Memorize** these expressions. Then verify that the probabilities you calculated in parts (b) and (c) satisfy

$$\sum_x f_X(x) = 1 \qquad \text{and} \qquad \sum_y f_Y(y) = 1$$

e. Explain in words why the sums in part (d) add up to one

4. Given the following double sum

$$\sum_x \sum_y f_{XY}(x,y)$$

a. Find this double sum for the joint probability distribution in Problem (3)

b. Explain why this double sum adds up to one

5. Given two random variables X and Y

a. What is the relationship between $f_{X|Y}(x|y)$ and $f_X(x)$ when X and Y are independent

b. What is the relationship between $f_{XY}(x,y)$, $f_X(x)$ and $f_Y(y)$ if X and Y are independent.

Hint - this is really the same relationship as that between $P(A \cap B)$, $P(A)$ and $P(B)$ when events A and B are independent

c. How could you determine if X and Y are independent from the values of $f_{XY}(x,y)$
d. Determine whether the random variables X and Y in Problem (3) are independent. How can you tell

6. Given the following probability distribution functions for the random variables X and Y

$f_X(1) = 0.3$ $\qquad$ $f_X(2) = 0.7$
$f_Y(1) = 0.6$ $\qquad$ $f_Y(2) = 0.4$

a. Find the joint probability distribution function $f_{XY}(x,y)$ if X and Y are independent. Explain how you got your result. Put your result in a Table
b. Repeat part (a) but this time come up with a possible joint probability distribution function $f_{XY}(x,y)$ if X and Y are not independent. Put your result in a Table

7. Generalizing on our definition for cumulative probability distributions of random variable X as follows

$$F_X(x) = P(X \le x)$$

we define **cumulative joint probability distribution functions** $F_{XY}(x,y)$ as follows

$$F_{XY}(x,y) = P(X \le x,\ Y \le y)$$

a. Calculate $F_{XY}(2,5) = P(X \le 2,\ Y \le 5)$ for the random variables X and Y with joint probability distribution $f_{XY}(x,y)$ as follows

Y \ X	1	2	3
3	0.1	0.2	0.1
4	0.1	0.1	0.1
5	0.1	0	0.2

b. Calculate $F_{XY}(x,y)$ for the random variables X and Y in part (a). Put your results in a Table like that for $f_{XY}(x,y)$
c. Describe and explain what's happening to the probabilities $F_{XY}(x,y)$ as we go down the columns and across the rows

8. Make use of the fact that

$$E[g(x,y)] = \sum_{x,y} g(x,y) f_{X,Y}(x,y)$$

to write out the sum of terms for calculating E[X + Y] if X has the values X = {2, 3}, Y has the values Y = {4, 5} and their joint probability distribution is given by $f_{XY}(x,y)$

9. Given that X and Y are the random variables with the following joint probability distribution $f_{XY}(x,y)$

Y \ X	2	3
4	0.3	0.4
5	0.2	0.1

a. Find E[X + Y]. Hint - set up and make use of a Table with columns X, Y, X + Y and $f_{XY}(x, y)$
b. Show that your result for E[X + Y] in part (a) is equal to E[X] + E[Y]
c. Find $E[X \ \ Y]$

10. Generalizing on the analysis in Problem (9b) it can be shown that in general

$$E[X_1 + X_2 + \ldots X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$$

Make use of this result to show that if $X_1, X_2, \ldots X_n$ are identically distributed random variables X then

$$E[X_1 + X_2 + \ldots X_n] = nE[X]$$

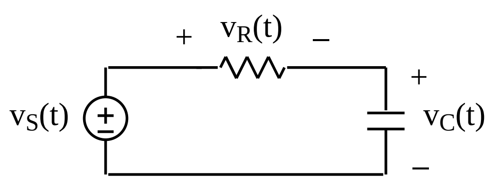
11. Make use of the result in Problem (10) to show that the expectation of a binomial random variable equal to the number of successes in n trials is given by E[X] = np

12. How are conditional probabilities as follows

$$f_{X|Y}(x,y)$$

different from joint probabilities. Illustrate with an example

13. Review Problem: Given the following circuit



Sketch the steady state sinusoids $v_S(t)$, $v_R(t)$ and $v_C(t)$ if the frequency f is a
a. Low frequency
b. High frequency