# ECE 307 - FOURIER TRANSFORMS - INVESTIGATION 9
# FROM FOURIER SERIES TO FOURIER TRANSFORM - PART II

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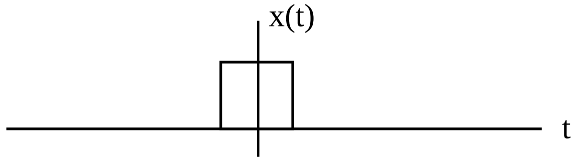
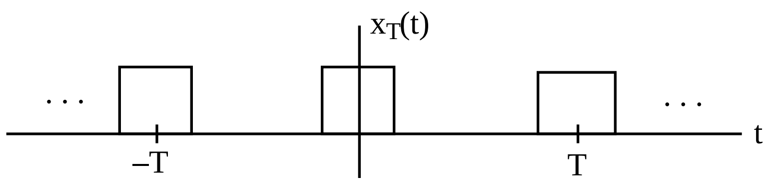
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From the last investigation we know that we can approximate the zero state response of a given linear circuit to a nonperiodic input like the following single pulse




by finding its response to the corresponding "constructed" periodic signals $x_T(t)$ as follows




as long as T is large enough for the transient response of the circuit to decay to zero between pulses. The objective of this investigation is to take the limit as $T \to \infty$ and get exact expressions for these zero state responses. What's going to happen is that our Fourier Series approximations of the input x(t) and zero state response y(t) as follows

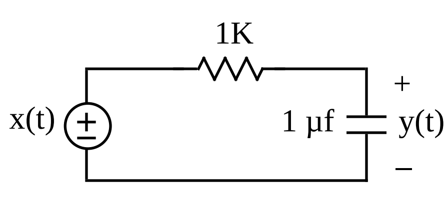
$$x_T(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t} \qquad \text{and} \qquad y_T(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_o t}$$

are going to metamorphize into the following integrals

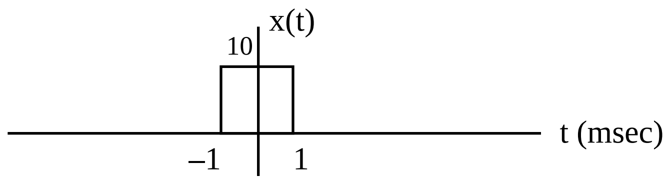
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad \text{and} \qquad y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega$$

as the discrete spectral components $X_k$ and $Y_k$ metamorphize into **continuous spectral densities** X(ω) and Y(ω). These spectral densities, as we will see, are very similar to other densities we work with all the time like mass densities, charge densities and probability densities.

1. As we said in the introduction we can use frequency domain analysis to approximate the zero state responses of linear circuits like the following

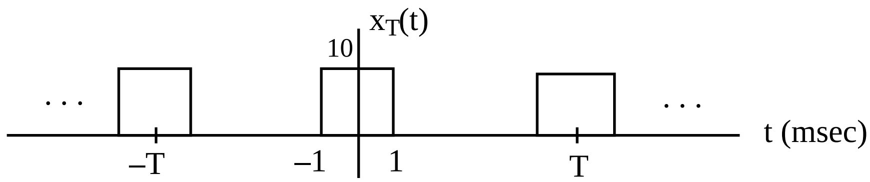


to inputs like the following pulse x(t)




by

(1) First **constructing** a corresponding periodic signal $x_T(t)$ as follows




with T large enough for the circuit's response to for all practical purposes decay to zero between pulses

(2) And then extracting from the steady state response

$$y_T(t) = \sum_{k=-\infty}^{\infty} Y_k \, e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} G(jk\omega_o) X_k \, e^{jk\omega_o t}$$

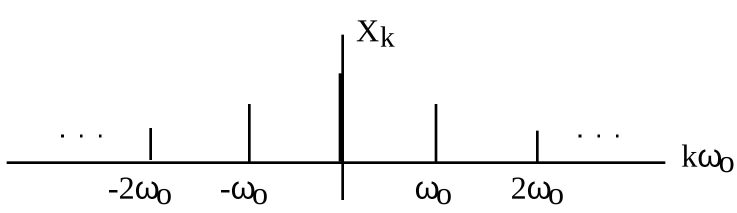
that part of the periodic response between $t = -1$ msec and $t = (T-1)$ msec

This scheme is great for approximating y(t). But to get an exact expression for y(t) – the integral that the sum

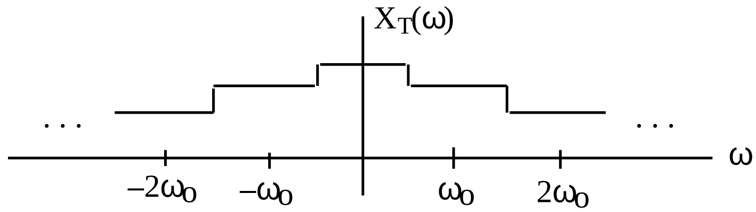
$$y_T(t) = \sum_{k=-\infty}^{\infty} Y_k \, e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} G(jk\omega_o) X_k \, e^{jk\omega_o t}$$

is approaching as $T \rightarrow \infty$, we first need to define what we mean by the **spectral density** $X(\omega)$ of a signal x(t).

We begin by first defining what we mean by the spectral density $X_T(\omega)$ of $x_T(t)$. The spectral density $X_T(\omega)$ of a constructed periodic signal $x_T(t)$ with period T and spectral plot as follows




is by definition a piecewise continuous signal as follows




with constant values in the intervals as follows

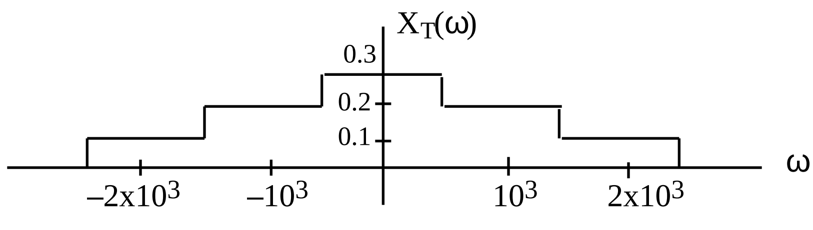
$$X_T(\omega) = \begin{cases} \vdots & \vdots \\ \dfrac{X_{-2}}{\omega_o/2\pi} = TX_{-2} & -2.5\omega_o \le \omega < -1.5\omega_o \\ \dfrac{X_{-1}}{\omega_o/2\pi} = TX_{-1} & -1.5\omega_o \le \omega < -0.5\omega_o \\ \dfrac{X_{0}}{\omega_o/2\pi} = TX_{0} & -0.5\omega_o \le \omega < 0.5\omega_o \\ \dfrac{X_{1}}{\omega_o/2\pi} = TX_{1} & 0.5\omega_o \le \omega < 1.5\omega_o \\ \dfrac{X_{2}}{\omega_o/2\pi} = TX_{2} & 1.5\omega_o \le \omega < 2.5\omega_o \\ \vdots & \vdots \end{cases}$$

From this we see that $X_T(\omega)$ is a **density** since its value in each interval is the value of the corresponding spectral component divided by a value proportional to the "distance" $\omega_o$ between spectral lines.

a. First find the constant values of the spectral density $X_T(\omega)$ in the frequency range $-2.5\omega_o < \omega < 2.5\omega_o$ if T = 1 msec and $X_o = 2$, $X_1 = 3e^{j1.2}$, $X_2 = e^{-j1.5}$.

b. Then use your results in part (a) to plot the magnitude $|X_T(\omega)|$ in this range.

2. From Problem (1) we know how to calculate and plot spectral densities $X_T(\omega)$ from spectral plots for $x_T(t)$. The objective of this problem is to illustrate how to regenerate the spectral plots of a signal $x_T(t)$ from its spectral density $X_T(\omega)$. Given the following spectral density




Find and sketch the corresponding spectral plot.

3. Now express the Fourier Series expansion for the constructed periodic signal $x_T(t)$ as follows

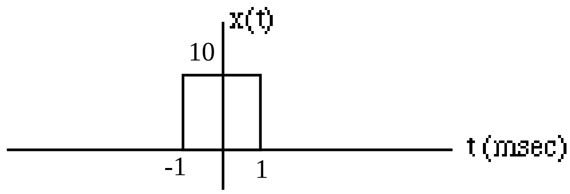
$$x_T(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

in terms of the spectral density $X_T(\omega)$.

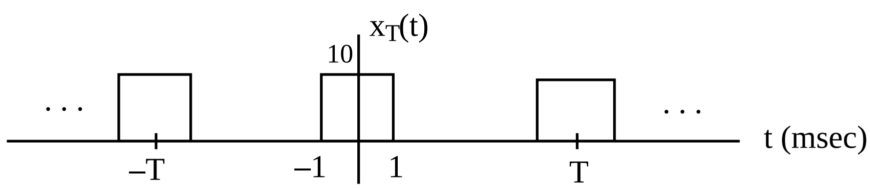
4. So far so good. We now define the spectral density $X(\omega)$ of a signal like a single pulse x(t) to be the limit as $T \rightarrow \infty$ of $X_T(\omega)$ as follows

$$X(\omega) \equiv \lim_{T \rightarrow \infty} X_T(\omega)$$

The objective of this problem is to get an idea of what the spectral density $X(\omega)$ of the following single pulse looks like




by seeing what happens to $X_T(\omega)$ as T gets larger and larger. From the corresponding constructed periodic pulse train given as follows




   a. First sketch the magnitude of the spectral density $X_T(\omega)$ for the first three lobes of harmonics when $T = 4$
   b. Then repeat part (a) for $T = 8$
   c. What's happening to the widths of the flat sections of the spectral density as T increases. How about the magnitudes.
   d. And now sketch what you expect $X(\omega)$ looks like for our single pulse

5. The objective of this problem is to come up with the integral equation for calculating the spectral densities $X(\omega)$ of signals like the pulse in Problem (4). We know from Problem (1) that the constant value of the spectral density $X_T(\omega)$ in the region around the k'th harmonic as follows

$$k\omega_o - \frac{1}{2}\omega_o \le \omega \le k\omega_o + \frac{1}{2}\omega_o$$

is given by

$$X_T(\omega) = \frac{X_k}{\omega_o / 2\pi} = TX_k = T\frac{1}{T}\int_T x_T(t)e^{-jk\omega_o t}dt = \int_T x_T(t)e^{-jk\omega_o t}dt$$

If we now take the limit as $T \to \infty$ then $x_T(t) \to x(t)$ and $X_T(\omega) \to X(\omega)$ with

$$X(\omega) \equiv \lim_{T \to \infty} X_T(\omega) = \lim_{T \to \infty} \int_T x_T(t) e^{-jk\omega_o t} dt = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

It's this **spectral density** $X(\omega)$ that we call the **Fourier Transform** of x(t) with

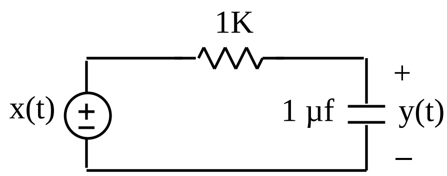
$$X(\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

**Memorize** this very important equation. Now for a specific example.

a. Calculate the Fourier Transform $X(\omega)$ of the pulse x(t) in Problem (1). Express your result in terms of the sinc function. Then plot $|X(\omega)|$. How is your plot related to your sketches in Problem (2)

b. How is your $X(\omega)$ in part (a) related to the envelope $X_{env}(\omega)$ of the spectral plot of $x_T(t)$

6. The objective of this problem is to show how the Fourier Transform of the output in a circuit like the following




is related to the Fourier Transform of the input x(t). From our Fourier Series results we know that if the input to a circuit is periodic then the Fourier coefficients of the output are given by

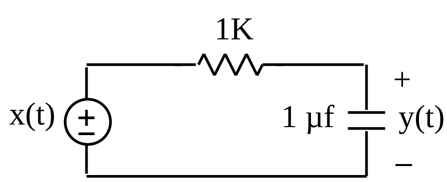
$$Y_k = G(jk\omega_o)X_k$$

Not surprisingly, the corresponding result for Fourier Transforms can be shown to be

$$Y(\omega) = G(j\omega)X(\omega)$$

where $X(\omega)$ is the Fourier Transform of the input and $Y(\omega)$ is the Fourier Transform of the output. **Memorize** this relationship. Then find $Y(\omega)$ for the zero state response of the circuit above if x(t) is the single pulse of Problem (1).

7. Now that we know how to find the Fourier Transform at the output of a circuit like the one below to an input like a single pulse we are finally in a position to get an exact expression for its zero state response



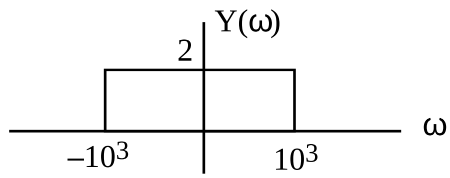

From our Fourier Series results we know that periodic signals $y_T(t)$ can be obtained from their Fourier coefficients $Y_n$ as follows

$$y_T(t) = \sum_{k=-\infty}^{\infty} Y_k\, e^{jk\omega_o t}$$

Not too surprisingly, we can derive the following analogous integral expression for y(t) given by

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\, e^{j\omega t}\, d\omega$$

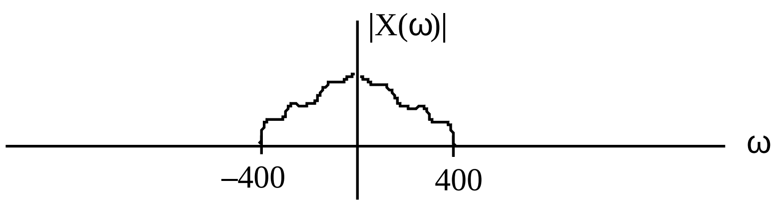
We refer to this expression as the **Inverse Fourier Transform** of Y(ω). Basically we're integrating the spectral density to get y(t) similarly to the way we integrate mass densities to get mass, probability densities to get probabilities and so on. The only difference is that we multiply by $e^{j\omega t}$ before taking the integral. **Memorize** this expression for the inverse Fourier Transform. Then make use of it to find and sketch y(t) for the following Y(ω)




8. Now suppose we have a lowpass circuit with transfer function

$$G(j\omega) = \frac{1000}{j\omega + 1000}$$

a. How would you expect the output y(t) to resemble the input x(t) for inputs with the following Fourier Transform. Explain how you got your answer – Include a sketch of |Y(ω)|




b. Repeat part (a) for the following signal