ECE 307 - COMPLEX FOURIER SERIES - INVESTIGATION 6
ENVELOPES OF SPECTRAL PLOTS

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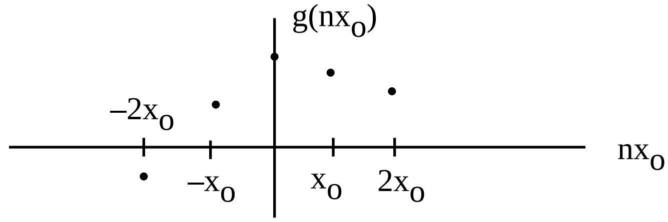
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

All our spectral plots of the magnitudes and phases of Complex Fourier Coefficients X_k have, of course, been discrete. The objective of this investigation is to introduce and make use of what we refer to as **envelopes** – the continuous curves going through the discrete points.

1. Let us begin with the simple function $g(x) = -2x^2 + 3$
   a. First make a discrete plot like the following



   of $g(nx_o) = -2(nx_o)^2 + 3$ as a function of nx_o for $x_o = 0.5$ and $n = -3, -2, \cdots, 2, 3$
   b. Now recopy your discrete plot from part (a) and then draw the continuous curve $g(x) = -2x^2 + 3$ going through your points. It's this continuous curve that we call the **envelope** of the discrete function $g(nx_o)$. We refer to it as $g_{env}(x)$.
   c. Describe in words how the expression $g_{env}(x)$ can be obtained from the expression for $g(nx_o)$ and vice versa.
   d. Make up an expression for $g(nx_o)$ and then find the corresponding envelope $g_{env}(x)$

2. The objective of this problem is to find the envelope for the **sinc** function defined as follows

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

   The sinc function is particularly important in electrical engineering - we see it all the time in spectral analysis of communications systems.
   a. First make a discrete spectral plot of $X(kf_o) = \mathrm{sinc}(kf_o)$ as a function of kf_o for $f_o = 0.2$ and $k = -20$ to $k = 20$. Make use of the fact that we *define* $\mathrm{sinc}(0) = 1$
   b. Now sketch the envelope $X_{env}(f)$ - the continuous curve going through the discrete points in your plot of part (a). Describe what the envelope looks like
   c. Find the equation for the envelope $X_{env}(f)$ of the sinc function
   d. In part (a) we make use of the fact that sinc(0) is defined to be 1. Show that

$$\lim_{x \to 0} \mathrm{sinc}(x) = 1$$

and so this definition is justified.

e. Show that if

$$X_{env}(f) = K \text{ sinc}(af)$$

then $X_{env}(0) = K$. **Memorize** this result.

3. The objective of this problem is to see how the value of *a* in sinc(af) affects the sinc function
   a. Sketch sinc(2f)
   b. Sketch sinc(5f)
   c. Make use of your results in parts (a) and (b) to describe the differences in the graphs of sinc(2f) and sinc(5f)

4. Sketch $X_{env}(f) = \text{sinc}(af)$ as a function of f. Then show that the **zero crossover frequencies** (the frequencies where $X(af) = 0$) are at integer multiples of 1/a. Note that we refer to each of the "bumps" as **lobes**

5. Show that the complex Fourier coefficients

$$X_k = X(kf_o) = \frac{ha}{T}\,\frac{\sin(\pi k f_o a)}{\pi k f_o a}\, e^{-j2\pi k f_o t_o}$$

of a general pulse train of magnitude *h*, pulse width *a*, period T and time delay t_o can be expressed in terms of the sinc function as follows

$$X_k = X(kf_o) = \frac{ha}{T}\,\text{sinc}(kf_o a)\, e^{-j2\pi k f_o t_o}$$

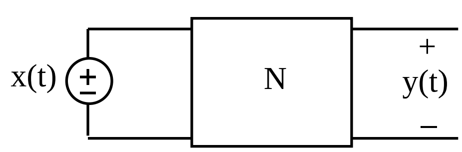
6. Make use of your result in Problem (5) to express the envelope $X_{env}(f)$ of the complex Fourier coefficients $X(kf_o) = X_k$ of a general pulse train in terms of the sinc function. Be **sure** to make a note of your result for future reference. Then find an expression for and sketch the magnitude of the envelope for $a = 10^{-3}$, $h = 5$, $T = 2\text{x}10^{-3}$ and $t_o = 0$

7. Write out the sum of the first three harmonics of x(t) in complex exponential form if

$$X_{env}(f) = 2 \text{ sinc}(f/2000)\, e^{-j2\pi f/5000} \qquad f_o = 1 \text{ KHz}$$

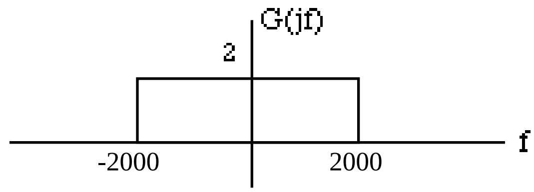
What is the average normalized power of the signal equal to your sum

8. Given the following circuit N

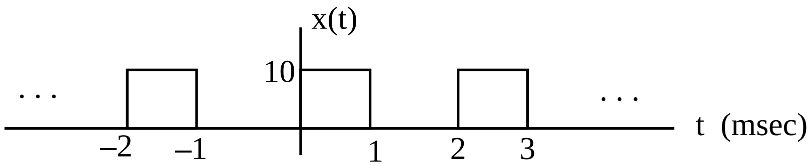


Make use of the fact that $Y(kf_o) = G(jkf_o)\,X(kf_o)$ where $X(kf_o) = X_k$ and $Y(kf_o) = Y_k$ to find an expression for $Y_{env}(f)$ in terms of G(jf) and $X_{env}(f)$. Describe how you got your result

9. Given a circuit N with frequency response as follows



and pulse train input



a. Sketch the magnitude of the spectral envelope $|X_{env}(f)|$ of the input x(t)
b. Sketch the magnitude of the spectral envelope $|Y_{env}(f)|$ of the output y(t)
c. Find the normalized average power of y(t)
d. What percentage of the x(t)'s average power reaches the output

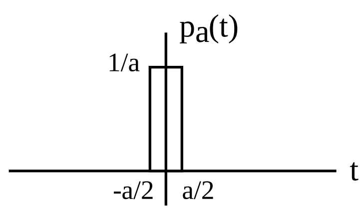
10. We'll be needing these results for the sinc function in the next several investigations

a. Show that $a \,\mathrm{sinc}\left(\frac{ax}{\pi}\right) = \frac{\sin(ax)}{x}$

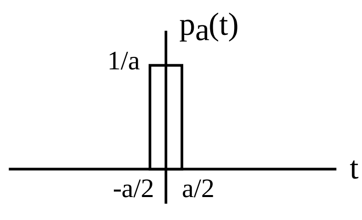
b. Show that $e^{j\omega a} - e^{-j\omega a} = 2j\omega a \,\mathrm{sinc}\left(\frac{\omega a}{\pi}\right)$

c. Sketch $X(f) = 2\,\mathrm{sinc}(af)$ for a = 1, 2 and 4. Describe what's happening to the widths and heights of the lobes as a increases
d. Sketch $X(f) = a\,\mathrm{sinc}(af)$ for a = 1, 2 and 4. Describe what's happening to the widths and heights of the lobes as a increases

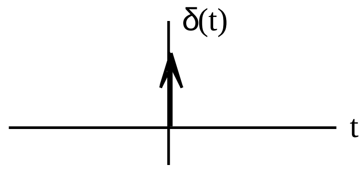
11. The objective of this problem is to introduce impulses δ(t). Impulses are - in effect - very narrow pulses of very large amplitude that have areas equal to one like the following pulse



when *a* is very small. More formally, the impulse function δ(t) is the *limit* as $a \to 0$ of our pulse of area one as follows



We represent impulses as follows



Impulses $\delta(t)$ are "characterized" by the fact that they are zero everyplace except the origin, they're infinite at the origin and they have areas equal to one. Impulses are clearly not "regular" functions like those in beginning calculus. But their properties are such that we can work with them just as if they were very narrow and tall pulses of area one. We use impulses extensively in developing important results. Sketch

a. $x_1(t) = \delta(t)$

b. $x_2(t) = \delta(t - 5)$

c. $x_3(t) = \delta(t + 5)$

d. $x_4(t) = \delta(t) + 2\delta(t - 3)$