ECE 307 - LAPLACE TRANSFORM - INVESTIGATION 15
SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH THE LAPLACE TRANSFORM - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

It was very straightforward to solve the first order linear differential equations of the last investigation when the solutions had LaPlace Transforms like

$$Y(s) = \frac{1000}{s + 2000}$$

But more generally our LaPlace Transforms are going to be rational polynomials like the following

$$Y(s) = \frac{s + 1000}{(s + 500)(s + 2000)}$$

A convenient way to obtain y(t) for such a rational polynomial is to make use of the result from algebra that rational polynomials can be written as a sum of terms as follows

$$Y(s) = \frac{s + 1000}{(s + 500)(s + 2000)} = \frac{K_1}{s + 500} + \frac{K_2}{s + 2000}$$

with constants K_1 and K_2. We call this a **partial fraction expansion**. The objective of this investigation is to come up with an algorithm for calculating K_1 and K_2 – even though we usually let computers do most of the actual computations. Once we have our partial fraction expansions we can then make use of our Table of LaPlace Transforms to easily find y(t)

1. There are several ways to obtain the coefficients K_1 and K_2 in the partial fraction expansion above. One convenient way is as follows
   a. Find K_1 by first multiplying both sides by (s + 500) and then setting s to –500.
   b. What happened to the term containing K_2 in the calculation of K_1 in part (a). Make use of this fact to justify that
   $$K_1 = (s + 500)Y(s)\Big|_{s=-500} = \frac{s + 1000}{s + 2000}\Bigg|_{s=-500}$$
   c. Find K_2
   d. Now use your results for K_1 and K_2 to find y(t)

2. Find and sketch y(t) satisfying y' + 100y = 50u(t) y(0) = 0

3. Find and sketch y(t) of the second order circuit with LaPlace Transform
   $$Y(s) = \frac{10^8}{s(s + 10^3)(s + 10^4)}$$
   having **real and different roots** in the denominator. Identify the natural and forced parts of the response. Is the natural response overdamped, underdamped or critically damped.

4. Now suppose we have a second order circuit with the following LaPlace Transform

$$Y(s) = \frac{10^8}{s(s+100+j1000)(s+100-j1000)}$$

having **complex conjugate roots** in the denominator

a. Explain how our method can be used to find the partial fraction expansion of Y(s)
b. How would you guess that the coefficients K_2 and K_3 of the terms corresponding to the complex conjugate roots are related. Why
c. Calculate the coefficients of the partial fraction expansion
d. Are the coefficients K_2 and K_3 related in the way you expected. If not, explain what's going on
e. Find and sketch y(t)
f. Identify the natural and forced parts of the response. Is the natural response overdamped, underdamped or critically damped.

5. If a circuit has a LaPlace Transform like the following

$$Y(s) = \frac{K}{(s+a)^2}$$

with two **real and equal roots** in the denominator then it can be shown that it has a partial fraction expansion as follows

$$Y(s) = \frac{K}{(s+a)^2} = \frac{K_1}{s+a} + \frac{K_2}{(s+a)^2}$$

with the corresponding y(t) equal to

$$y(t) = \left(K_1 e^{-at} + K_2 t e^{-at}\right) u(t)$$

Make use of these results to find and sketch y(t) with LaPlace Transform

$$Y(s) = \frac{10}{s+1000} + \frac{100}{(s+1000)^2}$$

Identify the natural and forced parts of the response. Is the natural response overdamped, underdamped or critically damped.

6. We know from this and the last investigation that if

$$Y(s) = \frac{Ks}{(s+p_1)(s+p_2)}$$

then y(t) will be of the form $K_1e^{-at} + K_2e^{-bt}$, $Ke^{-at}\cos(bt + \theta)$ or $K_1e^{-at} + K_2te^{-at}$ depending on whether p_1 and p_2 are real and different, complex conjugates or real and equal. Set up a Table that summarizes these relationships. Then **memorize** it