ECE 307 - LAPLACE TRANSFORM - INVESTIGATION 14
SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH THE LAPLACE TRANSFORM - PART I

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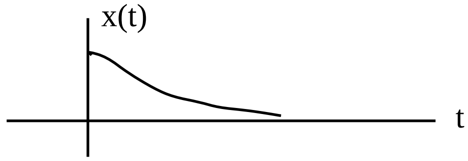
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this and the next couple of investigations is to show how we can use bilateral LaPlace Transforms to solve linear differential equations with constant coefficients and zero initial conditions

1. First show that the bilateral LaPlace Transform of the derivative of a signal x(t) is given by

$$L\left[\frac{d\,x(t)}{dt}\right] = sX(s)$$

   for signals x(t) that are zero for t < 0 like the following decaying exponential



   Hint – integrate by parts. **Memorize** this result for the **Bilateral LaPlace Transform**

2. Show that

$$L[a_1x_1(t) + a_2x_2(t)] = a_1X_1(s) + a_2X_2(s)$$

3. Now make use of your results in Problems (1) and (2) to solve for y(t) satisfying

$$y' + 500y = 2000\,\delta(t) \qquad y(0-) = 0$$

   by
   (1) First taking the bilateral LaPlace Transform of both sides of the equation
   (2) Then solving for Y(s)
   (3) And then finding the corresponding y(t) in the Table you put together in the previous investigation

   Explain why you were justified in using the result of Problem (1) when calculating the LaPlace Transform of the derivative of y(t)

4. Solve for y(t) satisfying

$$y' + 100y = 200\,\delta(t) \qquad y(0-) = 0$$

5. Let us now expand our Table of bilateral LaPlace Transforms
   a. First show that

$$2|K|\cos(\omega t + \angle K) = K\,e^{j\omega t} + K^*\,e^{-j\omega t}$$

   b. Then make use of your result in part (a) to show that

$$L\left[2|K|\cos(bt + \quad K)u(t)\right] = \frac{K}{s - jb} + \frac{K}{s + jb}$$

c. Make use of Euler's Relation to show that

$$L\left[2|K|e^{-at}\cos(bt + \quad K)u(t)\right] = \frac{K}{s + a - jb} + \frac{K}{s + a + jb}$$

d. Find $L\left[\frac{d^2x(t)}{dt^2}\right]$. Hint – make use of the fact that $L\left[\frac{d^2x(t)}{dt^2}\right] = L\left[\frac{d}{dt}\left(\frac{dx(t)}{dt}\right)\right]$

e. Show that $L\left[\int_{-\infty}^{t} x(t)\,dt\right] = \frac{1}{s}X(s)$. Hint – Let $w(t) = \int_{-\infty}^{t} x(t)\,dt$ and then take the derivative of both sides

f. Find the LaPlace Transform of the signal x(t) delayed by time t_d as follows

$$y(t) = x(t - t_d)\,u(t - t_d)$$

6. Find the LaPlace Transform of $x(t) = 10\,e^{-1000t}\cos(2000t + 1.2)\,u(t)$

7. Find Y(s) for y(t) satisfying the following differential equation

$$y'' + 1000y' + 10^6 y = 10^5 u(t)$$

with zero initial conditions

8. The objective of this and the next problem is to review material we are going to need in subsequent investigations. We know from ECE 207 that the complete responses of first and second order circuits to inputs like steps and sinusoids are a sum of a **natural response** and a **forced response**. We also know that the natural responses of such circuits are typically sums of decaying exponentials and damped sinusoids. And we know that the natural responses of second order circuits can be underdamped, overdamped or critically damped. Write out equations and then sketch graphs of natural responses that are
   a. Underdamped
   b. Overdamped
   c. Critically Damped

9. Sketch the step responses of i(t) and y(t) in the following underdamped circuit