ECE 307 - LAPLACE TRANSFORM - INVESTIGATION 13
FROM FOURIER TO LAPLACE TRANSFORMS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The Fourier results we obtained in the last several investigations are very useful in a number of applications - especially in communications and signal processing. But in other areas like analog circuit design and control system design, it's often more convenient to work with the **LaPlace Transform** - actually a variation on the Fourier Transform.

The objective of this investigation is to introduce the **Bilateral LaPlace Transform** and calculate it for some typical engineering signals. The Bilateral LaPlace Transform of a signal x(t) is calculated by simply taking the Fourier Transform of x(t) multiplied by the "scrunching" function $e^{-\sigma t}$ as follows

$$L[x(t)] = F[e^{-\sigma t} x(t)] = \int_{-\infty}^{\infty} e^{-\sigma t} x(t)\, e^{-j\omega t}\, dt$$

What's nice about multiplying x(t) by $e^{-\sigma t}$ is that the integrands now go to zero fast enough for typical engineering signals like constants and sinusoids that their LaPlace Transforms can be calculated with standard methods of first year calculus. This is nice not only because the math is simpler but also because the resulting Transforms do not contain impulses. **Memorize** the equation defining the Bilateral LaPlace Transform.

1. The objective of this first problem is to calculate the Bilateral LaPlace Transform of the unit step u(t).
   a. First sketch $u(t)e^{-\sigma t}$ for $\sigma > 0$, for $\sigma = 0$ and then for $\sigma < 0$. Describe how these signals differ.
   b. For what values of σ does the corresponding Bilateral LaPlace Transform Integral

   $$L\left[u(t)\right] = F\left[e^{-\sigma t}u(t)\right]$$

   exist.
   c. Calculate the Bilateral LaPlace Transform Integral in part (b) for those values of σ where the integral exists. Note that it is this function of σ and ω that we refer to as the Bilateral LaPlace Transform of u(t).

2. The objective of this problem is to calculate the Bilateral LaPlace Transform of $x(t) = e^{-1000t}\,u(t)$
   a. Sketch x(t)
   b. For what values of σ does the Bilateral LaPlaceTransform Integral

   $$L\left[e^{-1000t}u(t)\right] = F\left[e^{-\sigma t}e^{-1000t}u(t)\right]$$

   exist

c. Calculate the Bilateral LaPlace Transform of $x(t) = e^{-1000t}u(t)$ by calculating the corresponding integral for those values of σ where the integral exists.

3. Repeat Problem (2) for $x(t) = e^{-at}u(t)$ for real a.

4. Calculate the Bilateral LaPlace Transform for $x(t) = e^{-at}u(t)$ where a can be real or complex

5. Verify that the Bilateral LaPlace Transform of $\cos(bt)\,u(t)$ is equal to

$$\frac{1/2}{\sigma + j\omega - jb} + \frac{1/2}{\sigma + j\omega + jb} = \frac{1/2}{s - jb} + \frac{1/2}{s + jb} = \frac{s}{s^2 + b^2}$$

Hint – Make use of Euler's Relation. Note that we have replaced

$$\sigma + j\omega \text{ by } s$$

as we usually do.

6. Find the Bilateral LaPlace Transform of the impulse function δ(t). Remember that

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)\int_{-\infty}^{\infty} \delta(t)\,dt = f(0)$$

7. Set up a Table of the LaPlace Transforms L[x(t)] = X(s) of the functions x(t) we've been working with in this investigation.

8. How do you think your Table in Problem (7) might be useful in finding inverse LaPlace Transforms – finding the signal x(t) that has a given LaPlace Transform. Make up and do an example.