ECE 307 - FOURIER TRANSFORMS - INVESTIGATION 12
ENERGY AND POWER SPECTRAL DENSITIES

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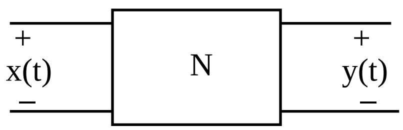
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From the last investigation we know that if x(t) is the input and y(t) the output of a linear circuit or system as follows



then

$$Y(\omega) = G(j\omega)\,X(\omega)$$

where $Y(\omega)$ and $X(\omega)$ are the Fourier Transforms of the input and output and $G(j\omega)$ is the frequency response of N. The objective of this investigation is to look at the powers and energies of these signals.

From Investigation 5 we know that if x(t) is periodic with period T then by Parseval's Theorem its average normalized power can be calculated in both the time and frequency domains as follows

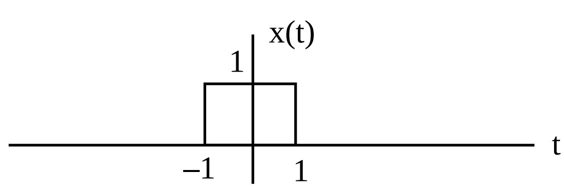
$$P_{av} = \frac{1}{T}\int_T x^2(t)\,dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

This relationship is particularly useful because it gives us a straightforward way to calculate the average normalized power of y(t) from $G(j\omega)$ and the spectrum of x(t) as follows

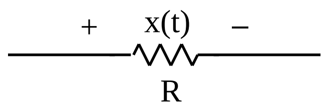
$$P_{av} = \frac{1}{T}\int_T y^2(t)\,dt = \sum_{k=-\infty}^{\infty} |Y_k|^2 = \sum_{k=-\infty}^{\infty} |G(jk\omega_o)X_k|^2 = \sum_{k=-\infty}^{\infty} |G(jk\omega_o)|^2 |X_k|^2$$

The objective of this investigation is to use our Fourier Transform results to come up with more general equations for energy and power that can be used for periodic as well as nonperiodic signals.

1. We begin with signals like the following pulse x(t)



Such signals are referred to as **energy signals** because when they're the voltages across resistors R as follows



then the total amount of energy delivered to the resistor is finite. Note that the **normalized energy** of an energy signal x(t) as follows

$$E = \int_{-\infty}^{\infty} p(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt$$

is by definition the total amount of energy that x(t) would deliver to a 1Ω resistor. **Memorize** this relationship. Then

a. Find the normalized energy of the pulse x(t) above
b. Find the normalized energy of $x(t) = e^{-1000t}u(t)$
c. Come up with an example of a signal that is not an energy signal. Explain how you know it is not an energy signal.

2. One particularly nice thing about energy signals like the pulse in Problem (1) is that they have Fourier Transforms and corresponding Inverse Fourier Transforms as follows

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

If we now substitute the equation for x(t) as follows

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

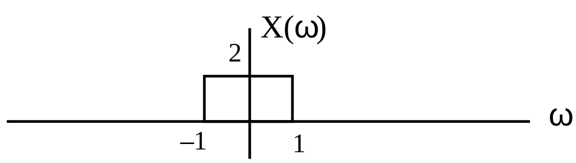
into our equation for the normalized energy E

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

and do the appropriate math then we'll obtain the following nice expression

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

for the normalized energy E as a function of the spectrum of x(t). **Memorize** this relation. And then make use of it to find the normalized energy of the signal x(t) with Fourier Transform as follows



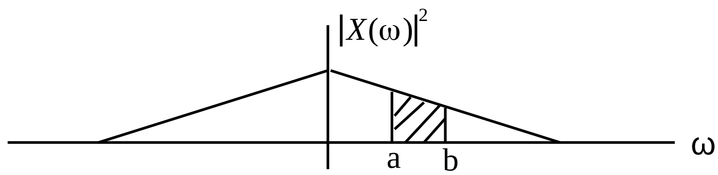
3. The objective of this problem is to define what we mean by Average Energy Spectral Density. From Problem (2) we know that the normalized energy of an energy signal x(t) is equal to the integral of $|X(\omega)|^2$ as follows

$$E = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

over the whole frequency range. If, however, we just integrate $|X(\omega)|^2$ over a part of the spectrum from $\omega = a$ to $\omega = b$ as follows

$$E_{ab} = \int_a^b |X(\omega)|^2 \, d\omega$$

then we get just that part of the normalized energy of x(t) in the frequency range (a, b) as illustrated in the following graph

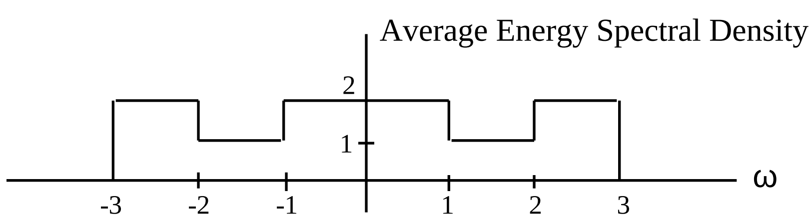


We now define the **Average Energy Spectral Density** in a given frequency interval (a, b) to equal

$$\text{Average Energy Spectral Density} = \frac{E_{ab}}{b-a} = \frac{\int_a^b |X(\omega)|^2 \, d\omega}{b-a}$$

in units of joules/(rad/sec)

a. Find the Average Energy Spectral Density of the energy signal x(t) in a given interval of the spectrum with b - a = 10 rad/sec and energy $E_{ab} = 0.001$ joules.
b. Find the normalized energy of the energy signal x(t) in a given interval of the spectrum with b - a = 5 rad/sec and Average Energy Spectral Density = 0.03 joules/(rad/sec)
c. Find the normalized energy of the energy signal x(t) with the following Average Energy Spectral Density in the units of joules/(rad/sec)



4. The objective of this problem is to define what we mean by Energy Spectral Density. From Problem (3) we know that the Average Energy Spectral Density of an energy signal x(t) in the frequency range (a, b) is given by

$$\text{Average Energy Spectral Density} = \frac{E_{ab}}{b-a} = \frac{\int_a^b |X(\omega)|^2 \, d\omega}{b-a}$$

Now if $a < \omega < b$ then the **Energy Spectral Density $E_x(\omega)$** of the energy signal x(t) at the frequency ω is by definition

$$E_x(\omega) = \lim_{b-a \to 0} \frac{\int_a^b |X(\omega)|^2 \, d\omega}{b-a}$$

But as $b-a \to 0$ then

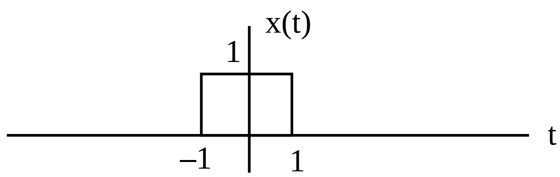
$$E_x(\omega) = \lim_{b-a\to 0} \frac{\int_a^b |X(\omega)|^2 d\omega}{b-a} \to \frac{|X(\omega)|^2 (b-a)}{(b-a)} = |X(\omega)|^2$$

and so the Energy Spectral Density of the energy signal x(t) is given by

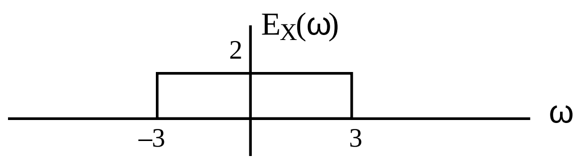
$$E_x(\omega) = |X(\omega)|^2$$

Memorize this relationship for energy signals. Then

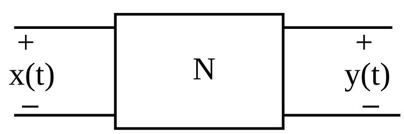
a. Find and sketch the energy spectral density of x(t) given by



b. Find the normalized energy of x(t) with Energy Spectral Density as follows

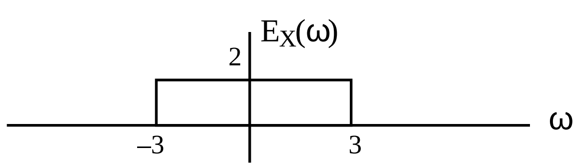


5. The objective of this problem is to find the energy spectral densities of the energy signals y(t) at the outputs of linear circuits. Given the following linear circuit N

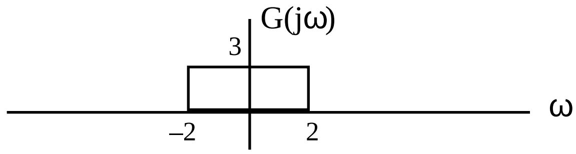


with input equal to the energy signal x(t)

a. Find an expression for the Energy Spectral Density of y(t) as a function of G(jω) and X(ω)
b. Find the energy of y(t) if x(t) has an energy spectral density as follows



and N has a frequency response given by



6. So far so good. For the rest of this investigation we're going to be working with power signals - signals whose total energy is infinite but whose

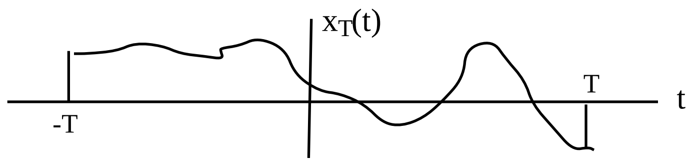
$$\text{A}verage[Energy\,/\,Sec] = \text{ Average [Power]}$$

is finite - those for which the following limit exists

$$P_{av} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)dt$$

   a. Explain why constants are power signals
   b. Explain why periodic signals like sinusoids and pulse trains are power signals
   c. Draw an example of a power signal that is not periodic
   d. Draw an example of a signal that is not a power signal

7. The objective of this problem is to define what we mean by the Power Spectral Densities $S_X(\omega)$ of power signals x(t). Power Spectral Densities are very similar to Energy Spectral Densities. The procedure for obtaining them is as follows
   (1) First truncate the power signal x(t) to $x_T(t)$ so it's an energy signal as follows



   (2) Then find the Energy Spectral Density of $x_T(t)$ as follows

$$|X_T(\omega)|^2$$

   (3) Then calculate the power spectral density of $x_T(t)$ over the time interval $-T \le t \le T$ as follows

$$\frac{1}{2T}|X_T(\omega)|^2$$

   (4) And finally take the limit as $T \to \infty$ and $x_T(t) \to x(t)$ to obtain the **Power Spectral Density $S_X(\omega)$** of x(t) as follows

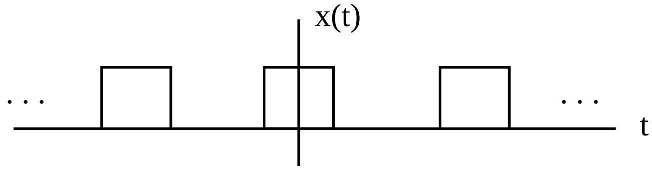
$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{2T}|X_T(\omega)|^2$$

   in units of watts/(rad/sec)

Come up with an integral equation for calculating the average normalized power P_{av} of a power signal in terms of its Power Spectral Density $S_X(\omega)$ analogous to our result for energy signals.

8. Sketch the power spectral density $S_X(\omega) = 2e^{-|\omega|/10}$ of x(t). Then make use of your result from Problem (7) $P_{av} = \int_{-\infty}^{\infty} S_x(\omega)\, d\omega$ to find the average power of x(t).

9. Let us now consider the special case of power signals that are periodic signals like the following pulse train



Verify that the power spectral density of such a signal

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j2\pi k f_o t}$$

as given by

$$S_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta\left(f - kf_o\right)$$

gives the correct value for the average power P_{av} of x(t)

10. Find and plot the power spectral density of

$$x(t) = 3 + 2\cos(2\pi 100t + 1.2) + \cos(2\pi 200t)$$

11. Make use of the result

$$S_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2\, \delta\left(f - kf_o\right)$$

to come up with an expression for the power spectral density $S_Y(f)$ at the output of a circuit in terms of its transfer function G(jf) and the spectrum X_k of its periodic input.