ECE 307 - FOURIER TRANSFORMS - INVESTIGATION 10
FOURIER TRANSFORMS OF SOME COMMON SIGNALS

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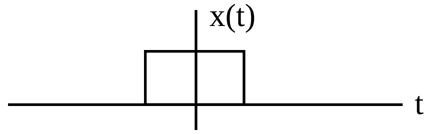
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The main result of the previous investigation is that the Fourier Transforms - the spectral densities - of signals like the following single pulse

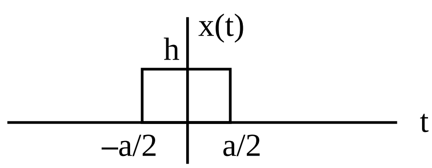


are given by the following integral

$$X(\omega) = F\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

The objective of this investigation is to extend these results to some of the basic signals we see in circuit and system analysis like constants, impulses and sinusoids.

1. Let us begin with following general pulse



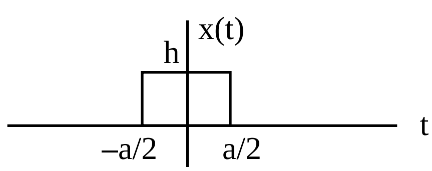
of magnitude h and pulse width a

a. Show that

$$X(\omega) = ha\,\text{sinc}\left(\frac{\omega a}{2\pi}\right)$$

b. Sketch $|X(\omega)|$

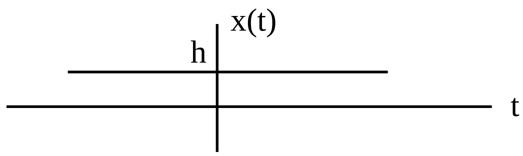
2. The objective of this problem is to find the Fourier Transform of a constant signal x(t) = h by taking the limit of the Fourier Transform of a pulse as follows



as $a \to \infty$

a. First sketch $|X(\omega)|$ for h = 2 and a = 1, 2 and 10

b. Describe what's happening to your graphs of $|X(\omega)|$ in part (a) as a increases. Your results should be consistent with the fact that as a increases and the pulse approaches a constant of value h as follows

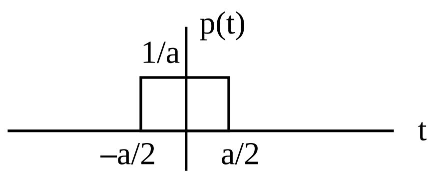


then the corresponding Fourier Transform X(ω) is approaching an impulse. In particular show that your results are consistent with the fact that

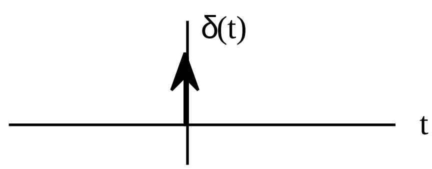
$$F[h] = \lim_{a \to \infty} X(\omega) = 2\pi h\,\delta(\omega)$$

c. Find and sketch the Fourier Transform of $x(t) = 10$

3. The objective of this problem is to see what happens to the Fourier Transform of the pulse in Problem (1) as we make it narrower and taller while keeping its area equal to one as follows



a. First sketch |X(ω)| for $a = 1, 0.5, 0.1$
b. Describe what's happening to your graphs of |X(ω)| as a decreases. Your results should be consistent with the fact that as $a \to 0$ and the pulse approaches an impulse as follows

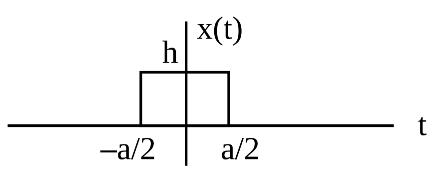


then the corresponding Fourier Transform X(ω) is approaching a constant. In particular show that your results are consistent with the fact that

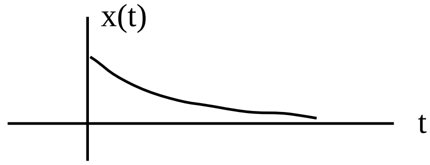
$$F[\delta(t)] = 1$$

Memorize this result.

4. In Problem (2) we found the Fourier Transform of a constant $x(t) = h$ by taking the limit of the Fourier Transform of a single pulse as follows



as a went to infinity. This is a standard trick for finding the Fourier Transforms of signals that go on forever. The objective of this problem is to apply this trick to the following decaying exponential as follows

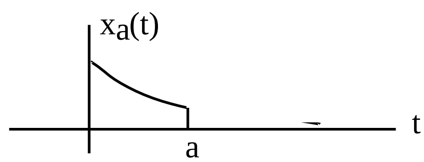


given by

$$x(t) = \begin{cases} Ke^{-bt} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{or equivalently} \quad x(t) = Ke^{-bt}u(t)$$

with b > 0.

a. First take the Fourier Transform $X_a(\omega)$ of x(t) truncated as follows

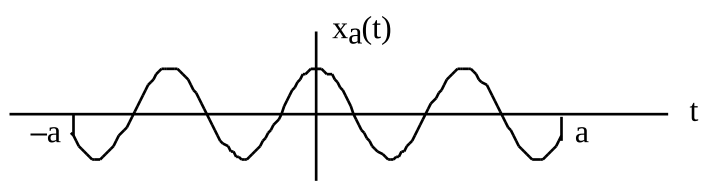


b. And then take the limit of $X_a(\omega)$ as $a \to \infty$ to obtain

$$X(\omega) = \lim_{a \to \infty} X_a(\omega)$$

Note that there are many functions for which the above limit does not exist – **not all functions x(t) have Fourier Transforms**.

5. Amazing as it may at first sound, we can also calculate Fourier Transforms of periodic signals like sinusoids using the same method we outlined in Problem (4)
   a. First make use of Euler's Relation to show that the Fourier Transform of the following truncated sinusoid $x_a(t)$ of frequency b as follows



is equal to

$$X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt = \int_{-a}^{a} \frac{e^{jbt} + e^{-jbt}}{2} e^{-j\omega t} dt$$

$$= a\left[\operatorname{sinc}\left(\frac{a(\omega - b)}{\pi}\right) + \operatorname{sinc}\left(\frac{a(\omega + b)}{\pi}\right)\right]$$

   b. Make a series of plots to illustrate what's happening to $X_a(\omega)$ as a gets larger. Describe and explain what's going on
   c. Verify that your graphical results are consistent with the fact that the Fourier Transform of a cosine is a sum of impulses as follows

$$F[\cos(bt)] = \lim_{a \to \infty} X_a(\omega) = \pi\delta(\omega - b) + \pi\delta(\omega + b)$$

6. Find the Fourier Transform of x(t) = 5 + 3 cos 100t + cos 200t

7. Make use of the following result

$$F\left[e^{jbt}\right] = 2\pi\delta(\omega - b)$$

to find the Fourier Transform of

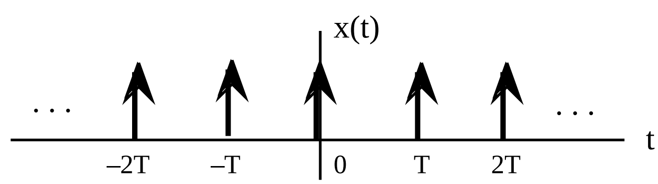
a. $x(t) = 3e^{-j1000t} + 4 + 3e^{j1000t}$

b. $x(t) = 3e^{j1.2}e^{-j1000t} + 4 + 3e^{-j1.2}e^{j1000t}$

8. Make use of the result in Problem (7) to come up with a general expression for the Fourier Transform of a periodic signal given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jk\omega_o t}$$

9. Find the Fourier Transform of an impulse train of period T as follows



Hint – you might find it helpful to make use of your result in Investigation 7 for the Fourier Series expansion of an impulse train. **Memorize** this result

10. Set up a Table of the Fourier Transforms developed in this investigation. Be sure to include Fourier Transforms of pulses, constants, impulses, decaying exponentials, sinusoids, pulse trains and impulse trains