ECE 307 - COMPLEX FOURIER SERIES - INVESTIGATION 1
SINUSOIDS AS SUMS OF COMPLEX EXPONENTIALS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In ECE 209 we showed how Fourier Series Expansions can be used to calculate steady state responses of linear circuits to periodic inputs. We referred to this way of analyzing circuits as **frequency domain** analysis because it involved first expressing the inputs as sums of sinusoids and then calculating responses by superposition - by adding up the responses to each of the individual sinusoids. The main objective of this class is to continue developing frequency domain methods to calculate not only steady state responses to inputs that are periodic but also transient responses to inputs that are periodic as well as nonperiodic. We will also introduce convolution and show how it is related to Fourier analysis.

The objective of this and the next several investigations is to make use of **Euler's Relation** as follows

$$re^{j\theta} = r\cos\theta + jr\sin\theta$$

to simplify and extend our results on Fourier Series analysis of periodic signals. What we'll be doing is very much like what we did in ECE 209 when we developed phasors except that now we're going to be expressing our sinusoids as sums of exponentials as follows

$$x(t) = 10\cos\left(10^3 t + \pi/4\right) = \frac{10e^{-j\pi/4}}{2}\, e^{-j10^3 t} + \frac{10e^{j\pi/4}}{2}\, e^{j10^3 t}$$

instead of as the real parts of complex exponentials like we've been doing as follows

$$x(t) = 10\cos(1000t + \pi/4) = \mathrm{Re}\left[10e^{j\pi/4}\, e^{j1000t}\right]$$

The main objective of this investigation is to express general Fourier Series expansions as follows

$$v(t) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t + \theta_k)$$

as sums of complex exponentials. Be sure to **memorize** Euler's Relation if you have not already done so.

1. First make use of Euler's Relation to show that $re^{j\theta}$ and $re^{-j\theta}$ are complex conjugates.

2. Then make use of Euler's Relation to show that x(t) as given in the introduction

$$x(t) = \frac{10}{2}\, e^{-j\pi/4}\, e^{-j10^3 t} + \frac{10}{2}\, e^{j\pi/4}\, e^{j10^3 t} = 5\, e^{-j(10^3 t + \pi/4)} + 5\, e^{j(10^3 t + \pi/4)}$$

   really is equal to $10\cos(1000t + \pi/4)$.

3. Now express each of the following as sums of sinusoids (and possibly a constant)
   a. $x(t) = 3\, e^{j\pi/3}\, e^{-j1000t} + 3\, e^{-j\pi/3}\, e^{j1000t}$
   b. $x(t) = 3\, e^{-j2000t} + 2\, e^{-j\pi/3}\, e^{-j1000t} + 4 + 2\, e^{j\pi/3}\, e^{j1000t} + 3\, e^{j2000t}$

4. Now go in the opposite direction and express each of the following signals x(t) as a sum of complex exponentials (and possibly a constant). Order the terms from lowest to highest frequency like in part (b) of Problem (3)
   a. $x(t) = 3 \cos(500t + \pi/3)$
   b. $x(t) = 2 \cos(100t + \pi/8) + 4 \cos(200t - \pi/4)$
   c. $x(t) = 3 + 2 \cos(100t + \pi/8) + 4 \cos(200t - \pi/4)$

5. Generalize on your results in Problem (4) to
   a. Express $x_1(t) = c_1 \cos(\omega_o t + \theta_1)$ as a sum of complex exponentials
   b. Express $x_k(t) = c_k \cos(k\omega_o t + \theta_k)$ as a sum of complex exponentials
   c. And now – to get more compact expressions – express your results in parts (a) and (b) in terms of

$$X_1 \equiv \frac{c_1}{2} e^{j\theta_1} \qquad X_{-1} \equiv \frac{c_1}{2} e^{-j\theta_1}$$

   and

$$X_k \equiv \frac{c_k}{2} e^{j\theta_k} \qquad X_{-k} \equiv \frac{c_k}{2} e^{-j\theta_k}$$

   d. What is the relation between X_1 and X_{-1} and more generally between X_k and X_{-k}. **Memorize** your result.

6. Now make use of your results in Problem (5) to show that the Fourier Series sum

$$x(t) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t + \theta_k)$$

   can be expressed in the nice **compact form**

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

   where

$$X_o = c_o \qquad X_{-k} = \frac{c_k}{2} e^{-j\theta_k} \qquad X_k = \frac{c_k}{2} e^{j\theta_k}$$

   Memorize this expression for x(t) as a sum of complex exponentials

7. We know from ECE 209 that the k'th harmonic $x_k(t)$ of a periodic signal x(t) can be expressed in terms of its phasor $X(jk\omega_o)$ as follows

$$x_k(t) = c_k \cos(k\omega_o t + \theta_k) = \mathrm{Re}\left[c_k e^{j\theta_k} e^{jk\omega_o t}\right] = \mathrm{Re}\left[X(jk\omega_o) e^{jk\omega_o t}\right]$$

   where $X(jk\omega_o) = c_k e^{j\theta_k}$. Make use of this result to first express X_k in terms of $X(jk\omega_o)$ and then express X_{-k} in terms of its complex conjugate $X^*(jk\omega_o)$. **Memorize** these results - you'll need them in Investigation 4.

8. Find X_{-2}, X_{-1}, X_0, X_1, X_2 for x(t) as follows

$$x(t) = 2 + 3 \cos(5t + 1.2) + 2 \cos(10t - 1.5)$$

9. Make use of Euler's Relation to find
   a. $e^{j\pi}$
   b. $e^{j2\pi}$
   c. $e^{j2\pi k}$

   Memorize these results.

10. Make use of the following result from ECE 209

$$e^{j(a+b)} = e^{ja}e^{jb}$$

to simplify the following expressions
   a. $e^{j(\theta+\pi)}$
   b. $e^{j(\theta+2\pi)}$
   c. $e^{j(\theta+2\pi k)}$

11. Make use of Euler's Relation to derive the following relation needed in the next investigation

$$\int e^{j\omega t} dt = \frac{1}{j\omega} e^{j\omega t}$$

12. Make use of Mathcad to obtain a graph of

$$x(t) = e^{j2t} + e^{-j2t}$$