ECE 306 - FREQUENCY RESPONSE - INVESTIGATION 9
FREQUENCY RESPONSES OF DISCRETE SYSTEMS - PART I

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As we've seen it's very straightforward to calculate the responses of discrete systems from their difference equations. We also know how to identify steady state responses to steps when the discrete systems are absolutely stable. And we know how to use convolution sums to calculate the responses of linear time-invariant discrete systems to arbitrary inputs. The objective of this Investigation is to develop a method for calculating sinusoidal steady state responses of linear time-invariant discrete systems analogous to what we developed in ECE 209 for linear analog circuits.

1. The objective of this first problem is to see how a stable linear time-invariant difference equation like the following

$$y[n] = 0.7y[n-1] + x[n]$$

responds to a sinusoidal input. Suppose, in particular, that $x[n]$ is the samples of $x(t) = 5\cos(2\pi 1000t)$ sampled at $f_s = 10$ KHz

a. Use the following Matlab program to obtain a stem graph of $x[n]$

```
f = 1e3;
fs = 1e4;
Ts = 1/fs;
N = 40;
n = 0: N;
x = 5*cos(2*pi*f*n*Ts);
stem (n, x)
title ('x[n] as a function of n')
xlabel ('n')
ylabel ('x[n]')
```

b. Use the following Matlab program to obtain a stem graph of $y[n]$ if $y[-1] = 0$

```
f = 1e3;
fs = 1e4;
Ts = 1/fs;
N = 40;
n = 0: N;
x = 5*cos(2*pi*f*n*Ts);
y = zeros (1, N+1);
y_1 = 0;
y(1) = 0.9*y_1 + x(1);
for n = 2: N+1
    y(n) = 0.9*y(n-1) + x(n);
end
n = 0: N
stem (n, y)
title ('y[n] as a function of n')
xlabel ('n')
ylabel ('y[n]')
```

c. Make use of your graph in part (b) to write an equation for the sinusoidal steady state response of $y[n]$ - the sinusoid left after the natural part of the response decays. Hint - draw a continuous curve through the $y[n]$ values.

d. How is the frequency of the steady state $y[n]$ related to the frequency of $x[n]$
e. How many samples did it take for your $y[n]$ to reach steady state - for the natural response to for all practical purposes decay to zero

2. Generalizing on the result of Problem (1) it can be shown that if

$$x[n] = A\cos\left(2\pi fnT_s + \theta\right)$$

is the input of a stable linear time-invariant difference equation like the following

$$y[n] = 0.5y[n-1] + 2x[n]$$

then the sinuoidal steady state response will be a sinusoid of the same frequency as follows

$$y[n] = B\cos\left(2\pi fnT_s + \phi\right)$$

just like for analog linear circuits and systems.

Our goal, as it was in the continuous case, is to find a nice "easy" way to calculate the amplitude B and phase φ of the sinusoidal steady state response. Again the trick is to make use of Euler's Relation as follows

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

from which we have

$$r\cos(\theta) = \mathrm{Re}\left[re^{j\theta}\right]$$

Make use of this relationship to express

a. The input $x[n] = A\cos\left(2\pi fnT_s + \theta\right)$ as the real part of a complex exponential
b. The sinusoidal steady state output $y[n] = B\cos\left(2\pi fnT_s + \phi\right)$ as the real part of a complex exponential

3. The objective of this problem is to make use of our results from the last problem as follows

$$x[n] = A\cos\left(2\pi fnT_s + \theta\right) = \mathrm{Re}\left[Ae^{j\theta}e^{j2\pi fnT_s}\right]$$
$$y[n] = B\cos\left(2\pi fnT_s + \phi\right) = \mathrm{Re}\left[Be^{j\phi}e^{j2\pi fnT_s}\right]$$

to help us calculate sinusoidal steady state responses. Suppose in particular that the input to the following difference equation

$$y[n] = 0.5y[n-1] + 2x[n]$$

is $x[n] = 5\cos\left(2\pi fnT_s + 1.2\right)$ with $f = 1\ KHz$ and $f_s = 4\ KHz$

a. Express $x[n]$ as the real part of a complex exponential
b. Substitute $x[n]$ and $y[n]$ into the difference equation to obtain

$$\mathrm{Re}\left[Be^{j\phi}e^{j\frac{\pi}{2}n}\right] = 0.5\,\mathrm{Re}\left[Be^{j\phi}e^{j\frac{\pi}{2}(n-1)}\right] + 2\mathrm{Re}\left[5e^{j1.2}e^{j\frac{\pi}{2}n}\right]$$

c. Now make use of the following two relationships for complex numbers

(1) $a\,\mathrm{Re}[z] = \mathrm{Re}[az]$ when a is real

(2) $a\,\mathrm{Re}\left[z_1\right] + b\,\mathrm{Re}\left[z_2\right] = \mathrm{Re}\left[az_1 + bz_2\right]$ when a and b are real

to show that $\mathrm{Re}\left[Be^{j\phi}\left(1-0.5e^{-j\frac{\pi}{2}}\right)e^{j\frac{\pi}{2}n}\right] = \mathrm{Re}\left[10e^{j1.2}e^{j\frac{\pi}{2}n}\right]$

d. And then make use of the third relationship

(3) If $\mathrm{Re}\left[z_1 e^{jbn}\right] = \mathrm{Re}\left[z_2 e^{jbn}\right]$ for all n then $z_1 = z_2$

to show that $Be^{j\phi} = \dfrac{10e^{j1.2}}{1-0.5e^{-j\frac{\pi}{2}}}$

e. Express $Be^{j\phi} = \dfrac{10e^{j1.2}}{1-0.5e^{-j\frac{\pi}{2}}}$ in complex exponential form

f. Make use of your result in part (e) to find the sinusoidal steady state response of $y[n]$ as a sampled sinusoid

4. Make use of the algorithm developed in Problem (3) to find the sinusoidal steady state response of the following difference equation

$$y[n] = 0.8y[n-1] - 3x[n]$$

to $x(t) = 2\cos(2\pi 1000t)$ sampled at $f_s = 3000$ samples/sec

5. Math Review: Find the integral of $x(t) = 4e^{j2\pi 100t}$ over one period

6. Match Review: Sketch the distance d between the two points in the complex plane as follows as a function of θ for $0 \le \theta \le 2\pi$