# ECE 306 - THE Z-TRANSFORM - INVESTIGATION 21
# SOLVING DIFFERENCE EQUATIONS WITH THE Z-TRANSFORM

**FALL 2006** **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we introduced the z-transform and then calculated it for some common signals. In particular we found that

$$Z\left[2|K|a^n \cos(bn + \quad K\right] = \frac{Kz}{z - ae^{jb}} + \frac{K^* z}{z - ae^{-jb}}$$

The objective of this Investigation is to use our results to find the complete responses of linear difference equations with constant coefficients to common signals like steps u[n].

1. We begin with the following difference equation

$$y[n] = 0.5y[n-1] + \delta[n] \qquad y[-1] = 0$$

   a. First find Y(z) by taking the z-transform of both sides of the equation
   b. Make use of your Table of z-transforms from the last Investigation to find $y[n]$
   c. Sketch $y[n]$

2. As we saw in the last problem, it's very straightforward to solve difference equations like the following

$$y[n] = 0.5y[n-1] + \delta[n]$$

   All we have to do is go to our Table and get the corresponding $y[n]$. But for more general z-transforms like the following ratio of polynomials in z

$$Y(z) = \frac{z^2}{(z+0.5)(z-0.7)}$$

   we need to make use of the result from algebra that such a rational polynomial can be written as a sum of terms as follows

$$Y(z) = \frac{z^2}{(z+0.5)(z-0.7)} = K_0 + \frac{K_1 z}{z+0.5} + \frac{K_2 z}{z-0.7}$$

   with constants $K_0, K_1$ and $K_2$. Note that these partial fraction expansions are very similar to the ones we use for LaPlace Transforms. Now there are several ways to obtain the coefficients $K_0, K_1$ and $K_2$. Go through the calculations for the following algorithm:

   a. Find $K_0$ by simply evaluating both sides of the equation with $z = 0$
   b. Find $K_1$ by multiplying both sides of the equation for $Y(z)$ by

$$\frac{z+0.5}{z}$$

   and setting $z = -0.5$. What happens to all the terms on the right hand side of the equation except the one for $K_1$

c. From part (b) we have that

$$K_1 = \frac{z+0.5}{z} Y(z)\Big|_{z=-0.5}$$

Make use of this scheme to find $K_2$

d. Make use of your results in part (a)-(c) to express $Y(z)$ as a sum of fractions and then make use of your Table of z-transforms in the previous Investigation to find an expression for $y[n]$

3. Find and sketch the **step response** - the response when

   (1) The input is the unit step: $x[n] = u[n]$

   (2) All initial conditions are zero

   of the difference equation

$$y[n] = -0.7y[n-1] + 2x[n]$$

4. Find the impulse response of the discrete system with difference equation as follows

$$y[n] = -0.5y[n-2] + 1.5y[n-1] + 2x[n]$$

5. Find the impulse response of the discrete system with difference equation as follows

$$y[n] = 1.3y[n-2] + 1.5y[n-1] + 2x[n]$$

6. Find and plot the zero state response of the following difference equation

$$y[n] = 0.5y[n-1] + 2x[n]$$

   with $x[n] = \cos(3n)u[n]$

7. Make use of the fact that if

$$Y(z) = z^{-1}W(z) \quad \text{then} \quad y[n] = w[n-1]$$

   to find $y[n]$ when

$$Y(z) = \frac{3}{z-0.5}$$

   Hint - note that $Y(z) = z^{-1}\left(\frac{3z}{z-0.5}\right)$

8. Find the impulse response - the zero state response - of the following difference equation when $x[n] = \delta[n]$

$$y[n] = 0.5y[n-1] + 2x[n-1]$$