ECE 306 - THE Z-TRANSFORM - INVESTIGATION 20
INTRODUCTION TO THE TWO-SIDED Z-TRANSFORM

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

Given a linear difference equation like the following

$$y[n] = 0.5y[n-1] - 0.3y[n-2] + x[n] + 2x[n-1]$$

with input $x[n]$ we know from our previous Investigations that

(1) The most straightforward way to calculate $y[n]$ for a particular value or values of n is to simply use a computer to evaluate the difference equation and

(2) The easiest way to calculate the sinusoidal steady state is with complex exponentials

The objective of this Investigation is to develop the *two-sided z-transform* as from the Discrete Time Fourier Transform (DTFT) to obtain closed form solutions of difference equations like

$$y[n] = 2(0.3)^n u[n] + 1.2u[n]$$

As we will see the z-transform is very similar to its analog cousin the LaPlace Transform and to the characteristic equations we used to calculate natural responses

1. In Investigation 17 we made use of the DTFT as follows

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

   to find the impulse response of a linear difference equation with constant coefficients. This scheme for finding complete responses works great for only for "nice" signals like pulses and impulses when the DTFT exists - when the infinite sum converges.

   The trick to getting the sum to converge for more general signals like constants $au[n]$ and sinusoids $a\cos(bn)u[n]$ is to multiply them by the decaying exponential $e^{-\Sigma n}$ as follows

$$x[n]e^{-\Sigma n}$$

   and then take the DTFT of these modified signals as follows

$$\sum_{n=-\infty}^{\infty} \left(x[n]e^{-\Sigma n}\right)e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-(\Sigma + j\Omega)n}$$

   Now by tradition we define

$$z \equiv e^{\Sigma + j\Omega}$$

   and then call the resulting sum from $-\infty$ to $+\infty$ as follows

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

the **two-sided or bilateral z-transform** of the discrete sequence $x[n]$. Note in particular that the z-transform is a complex function of the complex variable z. **Memorize** this definition. Then make use of it to find the z-transform of the sequence $x[n]$ equal to zero for all n except $x[0] = 1, x[1] = 3$ and $x[2] = -1$

2. The objective of this problem is to calculate the z-transforms of some simple sequences.
   a. Find the z-transform of the Kronecker Delta $\delta[n]$
   b. Show that the z-transform of the shifted Kronecker Delta $\delta[n - N]$ is equal to z^{-N}
   c. Show that the z-transform of the unit step $u[n]$ is as follows. Use a geometric sum

$$U(z) = \frac{z}{z-1}$$

3. Now show that the 2-sided z-transform, like the 2-sided LaPlace Transform, is linear - that it satisfies

$$Z\big[a_1 x_1[n] + a_2 x_2[n]\big] = a_1 Z\big[x_1[n]\big] + a_2 Z\big[x_2[n]\big]$$

4. Show that the z-transform of the geometric sequence x[n] = Kp^nu[n] is given by

$$Z\big[kp^n u[n]\big] = \frac{kz}{z-p}$$

5. Find the z-transform of

$$e^{-anT_s} u[n] = \text{samples of } e^{-at}u(t)$$

6. Show that the z-transform of the following sampled cosine is given by

$$Z\big[\cos(bn)u[n]\big] = \frac{1}{2}\,\frac{z}{z - e^{jb}} + \frac{1}{2}\,\frac{z}{z - e^{-jb}}$$

   Hint - Express the cosine as a sum of complex exponentials.

7. Generalize on your result from Problem (6) to find the z-transform of

$$x[n] = 2\,|K|\cos(bn + \quad K)u[n]$$

   Remember that $K = |K|e^{j \quad K}$ and $K \quad = |K|e^{-j \quad K}$

8. And then generalize on Problem (7) to find the z-transform of

$$x[n] = 2|K|a^n \cos(bn + \quad K)u[n]$$

9. Show that for the *two-sided or bilateral* z-transform

$$Z\big[x[n-m]\big] = z^{-m} X(z)$$

10. Put all your z-transforms from this Investigation into a Table