ECE 306 - FOURIER ANALYSIS - INVESTIGATION 17
INTRODUCTION TO DISCRETE TIME FOURIER TRANSFORM

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last three Investigations we showed how Discrete Time Fourier Series (DTFS) can be used to calculate steady state responses of linear time-invariant difference equations to discrete periodic inputs. Our main result was that if $x[n]$ is periodic with period N then we can write $x[n]$ as a finite sum of complex exponentials as follows

$$x[n] = \sum_{k=0}^{N-1} X_k e^{jk(2\pi/N)n}$$

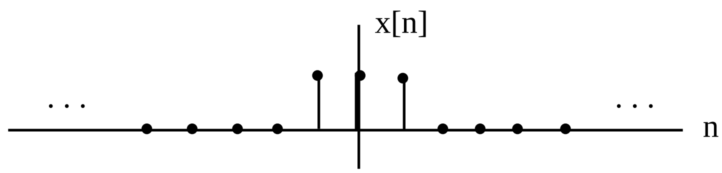
and then calculate the steady state output $\mathrm{y[n]}$ from

$$\mathrm{y[n]} = \sum_{k=0}^{N-1} \mathbf{Y}_k e^{jk(2\pi/N)n} = \sum_{k=0}^{N-1} H\left(e^{jk(2\pi/N)}\right) \mathbf{X}_k e^{jk(2\pi/N)n}$$

where $H\left(e^{j2\pi f T_s}\right)$ is the frequency response of the discrete system and the Discrete Time Fourier Series Coefficients $\mathbf{X}_k$'s are given by the finite sums

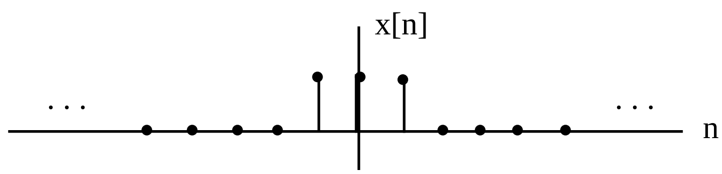
$$\mathbf{X}_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

More generally we would like to use digital signal processing to find the spectrums of nonperiodic signals from their samples $x[n]$ like the following

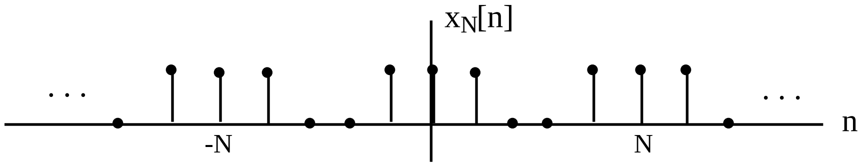


In this and the next Investigation we will introduce algorithms like the Discrete Time Fourier Transforms (DTFT) that can do this - but leave to a later course the details of how well the discrete spectrum of a set of samples $x[n]$ matches the spectrum of the corresponding continuous signal $x(t)$. We will also show how DTFT's can be used to calculate zero state responses. We start with finite duration discrete signals and then generalize.

1. To obtain the spectral density - the Fourier Transform - of a finite duration nonperiodic discrete signal like the following discrete pulse



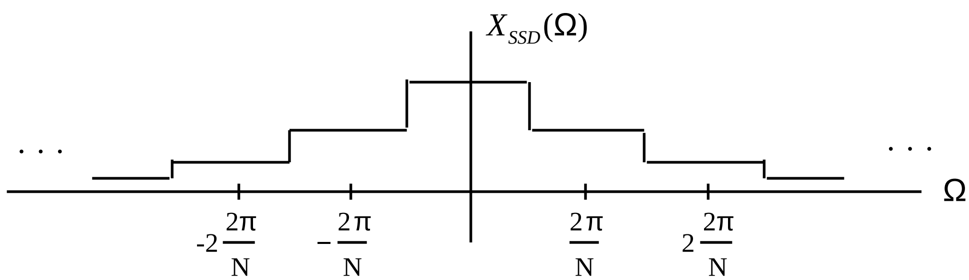
we start with the Discrete Time Fourier Series of a periodic signal $x_N[n]$ constructed from $x[n]$ as follows



and then take the limit as $N \to \infty$. This sounds "fine" but when we actually go to calculate the spectral components $\mathbf{X}_k$ of $x_N[n]$ as follows

$$\mathbf{X}_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} x[n]e^{-jk(2\pi/N)n} \le \frac{x[-1]+x[0]+x[1]}{N}$$

all the spectral coefficients $\mathbf{X}_k$ go to zero as we take the limit $N \to \infty$. But not so for the *staircase spectral density* $\mathbf{X}_{SSD}[\Omega]$ as follows



where the value of $\mathbf{X}_{SSD}[\Omega]$ in the k'th interval of width $2\pi/N$ is given by

$$\mathbf{X}_{SSD}[k] = \frac{\mathbf{X}_k}{1/N} = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} x[n]e^{-jk(2\pi/N)n}$$

Now as $N \to \infty$

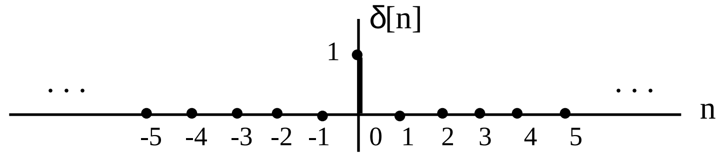
(1) discrete pulse train $\to$ discrete single pulse

(2) The flat stairs get narrower and narrower as $\mathbf{X}_{SSD}[\Omega]$ metamorphizes into the nice smooth *continuous* **Discrete Time Fourier Transform (DTFT)** of $x[n]$ given by

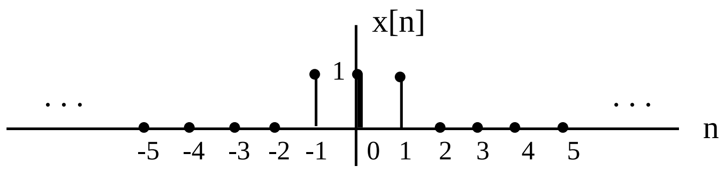
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Now make use of the equation for $X(\Omega)$ to find and sketch the DTFT's of the following finite duration sequences

a. The unit impulse $\delta[n]$



b. The single pulse



2. The objective of this problem is to show how $x[n]$ can be gotten back from its DTFT. The trick to doing this is to simply multiply both sides of

$$X(\Omega) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\Omega m}$$

by $e^{j\Omega n}$ and then integrate over 2π as follows

$$\int_{2\pi} X(\Omega) e^{jn\Omega} d\Omega = \int_{2\pi} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-jm\Omega} \right) e^{jn\Omega} d\Omega = \sum_{m=-\infty}^{\infty} x[m] \int_{2\pi} e^{j(n-m)\Omega} d\Omega = 2\pi x[n]$$

and so we have

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{jn\Omega} d\Omega$$

a. Verify the last step in the derivation - that

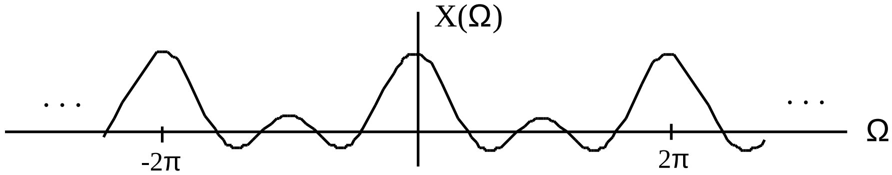
$$\frac{1}{2\pi} \int_{2\pi} e^{j(n-m)\Omega} d\Omega = \begin{cases} 1 & m = n \\ 0 & \text{otherwise} \end{cases}$$

b. Verify that this equation for $x[n]$ works for when $x[n] = \delta[n]$

3. Show that Discrete Time Fourier Transforms of sequences $x[n]$ as follows

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$$

are periodic of period 2π. In particular show that they satisfy $X(\Omega + 2\pi) = X(\Omega)$ like the following example does



4. The objective of this problem is to find the Discrete Time Fourier Transforms of delayed

sequences so we'll be able to find the Discrete Time Fourier Transforms of difference equations

a. Show that $DTFT[x[n-1]] = e^{-j\Omega}X(\Omega)$
b. Show that $DTFT[x[n-2]] = e^{-j2\Omega}X(\Omega)$
c. Generalize on (a) and (b) to come up with an expression for $DTFT[x[n-m]]$

5. Given the following difference equation

$$y[n] = 1.3\, y[n - 1] - 0.4\, y[n - 2] + x[n] + 1.5\, x[n - 1]$$

a. Make use of your results from Problem (4) to find $Y(\Omega)$ as a function of $X(\Omega)$
b. Now make use of your result in part (a) to find the transfer function

$$H\left(e^{j2\pi fT_s}\right) = H\left(e^{j\Omega}\right) = \frac{Y(\Omega)}{X(\Omega)}$$

6. Pulling together our results we have from Problem (5) that

$$Y(\Omega) = H\left(e^{j\Omega}\right)X(\Omega)$$

and so from Problem (2)

$$y[n] = \frac{1}{2\pi}\int_{2\pi} Y(\Omega)e^{jn\Omega}d\Omega = \frac{1}{2\pi}\int_{2\pi} H\left(e^{j\Omega}\right)X(\Omega)e^{jn\Omega}d\Omega$$

Now suppose we have the following difference equation with input $x[n] = \delta[n]$

$$y[n] = 1.3y[n-1] - 0.4y[n-2] + x[n] + 1.5x[n-1]$$

a. First find $y[0], y[1]$ and $y[2]$ of the zero state response directly from the difference equation
b. Then make use Matlab to calculate $y[0]$, $y[1]$ and $y[2]$ of the zero state response using our equation

$$y[n] = \frac{1}{2\pi}\int_{2\pi} Y(\Omega)e^{jn\Omega}d\Omega$$

with

$$Y(\Omega) = \frac{1 + 1.5e^{-j\Omega}}{1 - 1.3e^{-j\Omega} + 0.4e^{-j2\Omega}}\, X(\Omega) = H(e^{j\Omega})X(\Omega)$$

c. Confirm that your result in part (b) agrees with your result in part (a)

7. So far we've only been calculating the Discrete Time Fourier Transforms of finite duration sequences. More generally we define

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

to be the Discrete Time Fourier Transform of any sequence $x[n]$ for which the sum exists. Make use of this result to find the Fourier Transform of the sequence

$$x[n] = e^{-an}u[n] \qquad a > 0$$

8. MATLAB - Copy and run the following M-file

```
w = linspace (0, 2, 5)
y = 10.^w
x = logspace (0, 2, 5)
```

Make use of your results to determine

a. How many points are generated by linspace and logspace
b. Verify that x=y
c. Explain in words what's going on