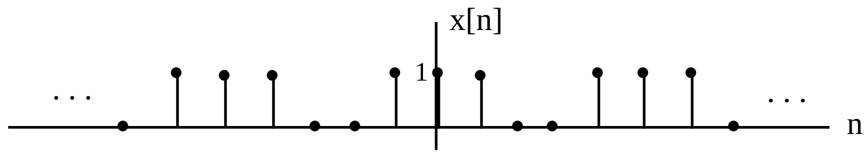
# ECE 306 - FOURIER ANALYSIS - INVESTIGATION 14
# DISCRETE TIME FOURIER SERIES - PART I

**FALL 2006** **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

From previous Investigations we know how to find the steady state responses of linear time-invariant difference equations to discrete steps, sinusoids and sums of sinusoids. The objective of this and the next Investigation is to first show that we can express periodic discrete signals $x[n]$ like the following




as sums of discrete sinusoids $x_k[n]$ as follows

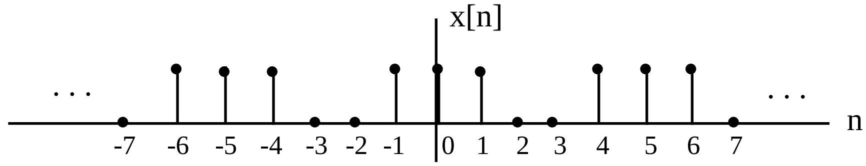
$$x[n] = x_1[n] + x_2[n] + \dots + x_N[n]$$

and then make use of superposition to find the steady state response to $x[n]$ as follows

$$y[n] = y_1[n] + y_2[n] + \dots + y_N[n]$$

where $y_1[n]$ is the sinusoidal steady state response to the discrete sinusoid $x_1[n]$ and so on. We refer to these finite sums of discrete sinusoids as Discrete Time Fourier Series (DTFS). They are the *spectrums* of the periodic signals.

1. The objective of this first problem is to formally define what we mean when we say a sequence $x[n]$ like the following is periodic




We say such a discrete sequence is **periodic** of period $N$ if for all $n$

$$x[n] = x[n+N]$$

   a. Find the period $N$ of $x[n]$ above
   b. Show that $x[n]$ satisfies $x[n] = x[n+N]$ for $n =$ 0, 1, 2, 7. Put your results in a Table.

2. Draw your own periodic sequence of period $N = 10$. Then verify that $x[n] = x[n+N]$ for $n =$ 2, 5, 7. Put your results in a Table.

3. The objective of this problem is to graph and find the period of the following periodic signal

$$x[n] = 1 - \cos\left(\left(\frac{2\pi}{2}\right)n\right)$$

a. Sketch $x[n]$
b. Show that $x[n]$ is periodic of period $N = 2$ by substituting $n + 2$ into the equation for $x[n]$ and then showing that $x[n+2] = x[n]$ for all $n$

4. Given the following sum of sinusoids

$$x[n] = 2 + \cos\left(\left(\frac{2\pi}{3}\right)n\right) - 1.5\cos\left(2\left(\frac{2\pi}{3}\right)n + 1.2\right)$$

a. Sketch $x[n]$
b. Show that $x[n]$ is periodic of period $N = 3$ by substituting $n + 3$ into the equation for $x[n]$ and then showing that $x[n+3] = x[n]$ for all $n$

5. Generalizing on the results of Problems (3) and (4) it can be shown that if $x[n]$ is a finite sum of sinusoids at the frequencies

$$0\,,\ \left(\frac{2\pi}{N}\right),\ 2\left(\frac{2\pi}{N}\right),\ 3\left(\frac{2\pi}{N}\right),\ \cdots\,,\ (N-1)\left(\frac{2\pi}{N}\right)$$

as follows

$$x[n] = c_o + \sum_{k=1}^{N-1} c_k \cos\left(k\left(\frac{2\pi}{N}\right) + \theta_k\right)$$

then $x[n]$ is periodic of period $N$. The objective of this problem is to demonstrate how Euler's Relation as follows

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

can be used to express sums of sinusoids as sums of complex exponentials

a. First make use of Euler's Relation to show that

$$r\cos(\theta) = \frac{re^{j\theta} + re^{-j\theta}}{2}$$

b. Now make use of your result in part (a) to express the following sum of discrete sinusoids as a sum complex exponentials

$$x[n] = 2 + \cos\left(\left(\frac{2\pi}{3}\right)n\right) + 2\cos\left(2\left(\frac{2\pi}{3}\right)n + 1.2\right)$$

6. The objective of this problem is to take the result from Problem (5) as follows

$$x[n] = 2 + \cos\left(\left(\frac{2\pi}{3}\right)n\right) + 2\cos\left(2\left(\frac{2\pi}{3}\right)n + 1.2\right) = \sum_{k=-2}^{2} d_k e^{jk(2\pi/3)n}$$

with $x[n]$ equal to a sum of complex exponentials from $k = -2$ to $k = 2$ and show that it can be written as a sum of complex exponentials from $k = 0$ to $k = 2$ as follows

$$x[n] = 2 + \cos\left(\left(\frac{2\pi}{3}\right)n\right) + 2\cos\left(2\left(\frac{2\pi}{3}\right)n + 1.2\right) = \sum_{k=0}^{2} X_k e^{jk(2\pi/3)n}$$

a. First show that $e^{j\theta} = e^{j(\theta + 2\pi n)}$
b. Then make use of your result in part (a) to show that

$$x[n] = 2 + \cos\left(\left(\frac{2\pi}{3}\right)n\right) + 2\cos\left(2\left(\frac{2\pi}{3}\right)n + 1.2\right)$$

is equal to

$$x[n] = 2 + \left(0.5 + e^{-j1.2}\right)e^{j\left(\frac{2\pi}{3}\right)n} + \left(0.5 + e^{j1.2}\right)e^{j2\left(\frac{2\pi}{3}\right)n}$$

7. Generalizing on the results of Problems (5) and (6) it can be shown that every sum of discrete sinusoids as follows

$$x[n] = c_o + \sum_{k=1}^{N-1} c_k \cos\left(k\left(\frac{2\pi}{N}\right)n + \theta_k\right)$$

can be expressed as a finite sum of complex exponentials from $k = 0$ to $k = N-1$ as follows

$$x[n] = c_o + \sum_{k=1}^{N-1} c_k \cos\left(k\left(\frac{2\pi}{N}\right)n + \theta_k\right) = \sum_{k=0}^{N-1} X_k e^{jk(2\pi/N)n}$$

The objective of this and the next several problems is to show the *converse* - the fact that if $x[n]$ is periodic of period $N$ then $x[n]$ can be expressed as a finite sum of complex exponentials - or equivalently discrete sinusoids - as follows

$$x[n] = \sum_{k=0}^{N-1} X_k e^{jk(2\pi/N)n}$$

Note that we refer to this sum as a **Discrete Time Fourier Series (DTFS)**. Our goal is to come up with equations for finding the $X_k$'s for arbitary periodic $x[n]$'s. But before we can do this we need to establish some of the basic properties of discrete complex exponentials $e_k[n]$ as follows

$$e_k[n] = e^{jk(2\pi/N)n}$$

a. First show that $e_k[n]$ is periodic in k with period $N$. In particular, show that

$$e_{k+mN}[n] = e_k[n]$$

for all integers $m$

b. Now make use of the equation for a finite geometric sum

$$S = \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \qquad a \neq 1$$

to verify that $e_k[n]$ satisfies the following **orthogonality** relationship

$$\sum_{n=0}^{N-1} e_k[n] = \sum_{n=0}^{N-1} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N & k = mN \quad \text{(k a multiple of N)} \\ 0 & k \neq mN \quad \text{(k not a multiple of N )} \end{cases}$$

for all integers m with $a = e^{jk(2\pi/N)}$

8. The objective of this problem is to make use of our result from Problem (7) as follows

$$\sum_{n=0}^{N-1} e_k[n] = \sum_{n=0}^{N-1} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N & k = mN \quad \text{(k a multiple of N)} \\ 0 & k \neq mN \quad \text{(k not a multiple of N )} \end{cases}$$

to find formulas for the $\mathbf{X}_k$'$s$ in terms of the $x[n]$'$s$. Suppose in particular that we have a periodic sequence $x[n]$ of period $N = 3$ expressed as follows

$$x[n] = X_0 + X_1 e^{j(2\pi/3)n} + X_2 e^{j2(2\pi/3)n}$$

a. Find an expression for $\mathbf{X}_0$ in terms of the $x[n]$'$s$ by taking the sum of both sides of the equation from $n = 0$ to $n = 2$ as follows

$$\sum_{n=0}^{2} x[n] = \sum_{n=0}^{2} \left( X_0 + X_1 e^{j(2\pi/3)n} + X_2 e^{j2(2\pi/3)n} \right)$$

and then make use of the result from Problem (7) that

$$\sum_{n=0}^{2} X_1 e^{j(2\pi/3)n} = \sum_{n=0}^{2} X_2 e^{j2(2\pi/3)n} = 0$$

b. Find an expression for $\mathbf{X}_1$ by multiplying both sides of the equation for by $e^{-j(2\pi/3)n}$ and then taking the sum of both sides from $n = 0$ to $n = 2$ as follows

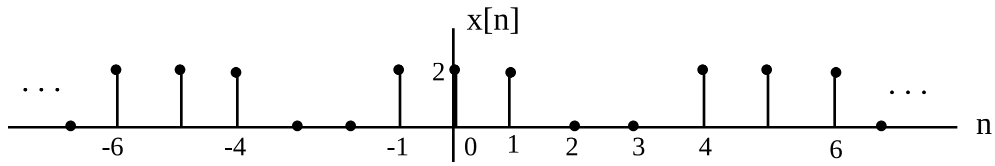
$$\sum_{n=0}^{2} x[n] e^{-j(2\pi/3)n} = \sum_{n=0}^{2} \left( X_0 e^{-j(2\pi/3)n} + X_1 + X_2 e^{j(2\pi/3)n} \right)$$

c. Similarly find an expression for $\mathbf{X}_2$

9. Generalizing on the results of Problem (8) we have that if $x[n]$ is a periodic sequence of period $N$ then $x[n]$ can be expressed as a sum of complex exponentials with discrete Fourier Series Coefficients $\mathbf{X}_k$ as follows

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n}$$

**Memorize** this result. And then make use of it to find and plot the magnitude of the discrete Fourier Series Coefficients $\mathbf{X}_k$'$s$ of the following periodic sequence




10. Given the following periodic signal $x(t) = 5\cos(2\pi 1000 t)$ sampled at $f_s = 5000$ samples/sec

a. Find the discrete Fourier Series Coefficients $X_k$
b. Draw the spectral plot of the magnitudes of the $X_k$'s from $k = 0$ to $k = 6$

11. MATLAB - Copy and run the following M-file

```
x1 = ones(1, 10);   x2 = zeros(1, 10);
y = [x1 x2];
n = 0 : 19;
plot (n, y)
```

Describe what this program does and how it does it