ECE 306 - FREQUENCY RESPONSE - INVESTIGATION 12
FREQUENCY RESPONSES OF DISCRETE SYSTEMS - PART IV

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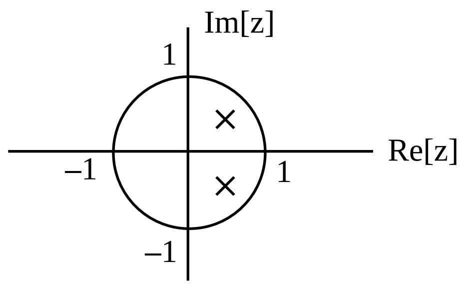
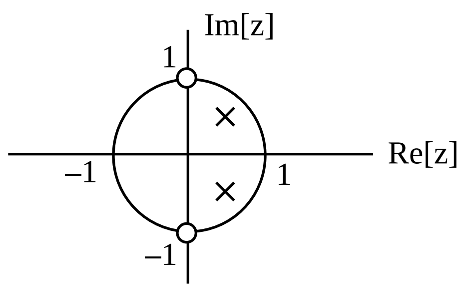
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In Investigation 8 we showed how the locations of the characteristic roots in a *characteristic root diagram* like the following



is related to the shape of the natural response of a linear time-invariant difference equation. The objective of this and the next Investigation is to extend these results to what are called *pole-zero diagrams* like the following



from which we can ascertain not only the shape of natural response but also the shape of frequency response $H\left(e^{j2\pi fT_s}\right)$. These relationships turn out to be extremely helpful not only in analysis but also in design.

1. We begin with a review problem. Find the steady state response of the following linear time-invariant difference equation

$$y[n] = 0.4y[n-1] + 0.7y[n-2] + x[n]$$

   to $x(t) = 5\cos(2\pi 2000t + 1.2)$ sampled at $f_s = 6000$ samples/sec

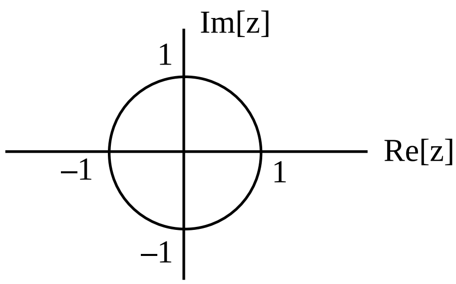
2. From the last three Investigations we know how to find and use frequency responses $H\left(e^{j2\pi fT_s}\right)$ to calculate sinusoidal steady state responses. The objective of this problem is to define what we mean by the **poles and zeros** of these transfer functions. The *zeros are simply the roots of the numerator and the poles the roots of the denominator* of the transfer function $H(z)$ obtained by replacing every $e^{j2\pi fT_s}$ in $H\left(e^{j2\pi fT_s}\right)$ by z as follows

$$H(z) = H\left(e^{j2\pi fT_s}\right)\Big|_{e^{j2\pi fT_s} \to z}$$

In particular suppose that

$$H\left(e^{j2\pi fT_s}\right) = \frac{e^{j2\pi fT_s} - 1}{e^{j2\pi fT_s} + 0.5}$$

a. Find $H(z)$
b. Find the zeros $z_1, z_2, \ldots$ equal to the roots of the numerator of $H(z)$
c. Find the poles $p_1, p_2, \ldots$ equal to the roots of the denominator of $H(z)$
d. Plot the poles and zeros in the complex plane as follows



with O's for the zeros and X's for the poles. Note that we refer to these plots as **pole-zero diagrams**. **Memorize** this term.

3. Given the following difference equation

$$y[n] - 0.4y[n-1] + 0.8y[n-2] = x[n]$$

a. Find the frequency response $H\left(e^{j2\pi fT_s}\right)$
b. Find the corresponding transfer function $H(z)$
c. Find the zeros $z_1, z_2, \ldots$
d. Find the poles $p_1, p_2, \ldots$
e. Draw the pole-zero diagram

4. Generalizing on the results of the last two problems we see that we can express $H(z)$ in terms of its poles and zeros as follows

$$H(z) = K\frac{(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

where K is a constant. Given a transfer function $H(z)$ with $K = 2$, zero $z_1 = -1$ and poles $p_1, p_1 = -0.6 \pm j0.5$

a. Sketch the pole-zero diagram
b. Find $H(z)$
c. Find $H\left(e^{j2\pi fT_s}\right)$

5. The objective of this and the rest of the problems in this Investigation is to show how the locations of the poles and zeros of $H(z)$ are related to the shape of its frequency response $H\left(e^{j2\pi fT_s}\right)$. We begin by showing how $H\left(e^{j2\pi fT_s}\right)$ can be expressed in terms of its poles and zeros. Given the following transfer function

$$H(z) = 2\frac{z(z+0.5)}{(z-0.3+j0.6)(z-0.3-j0.6)}$$

a. Find the poles and zeros
b. Sketch the pole-zero diagram
c. Find $H\left(e^{j2\pi fT_s}\right)$ in terms of its poles and zeros by simply replacing every z by $e^{j2\pi fT_s}$ as follows

$$H(e^{j2\pi fT_s}) = H(z)\big|_{z\to e^{j2\pi fT_s}}$$

6. Generalize on your result in Problem (5) to obtain $H(e^{j2\pi fT_s})$ as a function of its poles and zeros when

$$H(z) = K\frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

7. We now need to review some basic properties of complex number expressed in complex exponential form as follows

$$z_1 = r_1e^{j\theta_1} \quad \text{and} \quad z_2 = r_2e^{j\theta_2}$$

a. Find $|z_1|$ and $|z_2|$
b. Verify that $|z_1z_2| = |z_1||z_2|$
c. Verify that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

8. The objective of this problem is to review the geometric meaning of the absolute value of the difference between two complex numbers as follows

$$|z_1 - z_2|$$

a. First calculate $|z_1 - z_2|$ with $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$
b. Now make use of your result in part (a) to show that

$$|z_1 - z_2| = \text{ distance between } z_1 \text{ and } z_2$$

Draw a picture in the complex plane that illustrates what's going on.

9. Making use of our result in Problem (8) we see that

$$\left|e^{j2\pi fT_s} - z\right| = \text{ distance between } e^{j2\pi fT_s} \text{ and } z$$

a. Show that $e^{j2\pi fT_s}$ is on the unit circle
b. Draw a picture of the complex plane showing where $e^{j2\pi fT_s}$ is located for $2\pi fT_s = 0$, $\pi/4$, $\pi/2$, π, $3\pi/2$
c. Find f as a function of f_s when $2\pi fT_s = \pi$

d. Locate $e^{j2\pi fT_s}$ with $2\pi fT_s = \pi/4$ and z = -0.5 + j0.5 in the complex plane and then draw the line between them of length $\left|e^{j2\pi fT_s} - z\right|$

10. Pulling together the results of the last four problems we have that

$$H(e^{j2\pi fT_s}) = K\left|\frac{(e^{j2\pi fT_s} - z_1)(e^{j2\pi fT_s} - z_2)\cdots(e^{j2\pi fT_s} - z_m)}{(e^{j2\pi fT_s} - p_1)(e^{j2\pi fT_s} - p_2)\cdots(e^{j2\pi fT_s} - p_n)}\right|$$

$$= K\frac{\left|e^{j2\pi fT_s} - z_1\right|\left|e^{j2\pi fT_s} - z_2\right|\cdots\left|e^{j2\pi fT_s} - z_m\right|}{\left|e^{j2\pi fT_s} - p_1\right|\left|e^{j2\pi fT_s} - p_2\right|\cdots\left|e^{j2\pi fT_s} - p_n\right|}$$

$$\left|H\left(e^{j2\pi fT_S}\right)\right| = K\frac{DZ1\left(e^{j2\pi fT_S}\right)\cdot\ DZ2\left(e^{j2\pi fT_S}\right)\cdot\ \cdots\ DZm\left(e^{j2\pi fT_S}\right)}{DP1\left(e^{j2\pi fT_S}\right)\cdot\ DP2\left(e^{j2\pi fT_S}\right)\cdot\ \cdots\ DPn\left(e^{j2\pi fT_S}\right)}$$

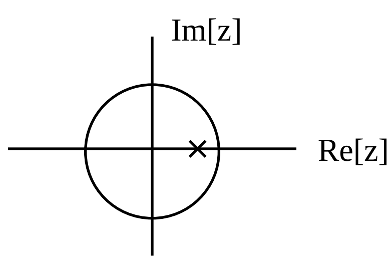
where

(1) $DZk\left(e^{j2\pi fT_S}\right)$ is equal to the distance between $e^{j2\pi fT_S}$ and the zero z_k

(2) $DPk\left(e^{j2\pi fT_S}\right)$ is equal to the distance between $e^{j2\pi fT_S}$ and the pole p_k

And so $\left|H\left(e^{j2\pi fT_S}\right)\right|$ is proportional to the product of the distances between $e^{j2\pi fT_S}$ and the zeros $z_1, z_2, \ldots, z_m$ divided by the product of the distances between $e^{j2\pi fT_S}$ and the poles $p_1, p_2, \ldots, p_n$. Make use of this result to sketch the magnitudes of the frequency responses corresponding to each of the following pole-zero diagrams. In each case indicate whether the discrete system is lowpass, bandpass, highpass or bandstop

a.



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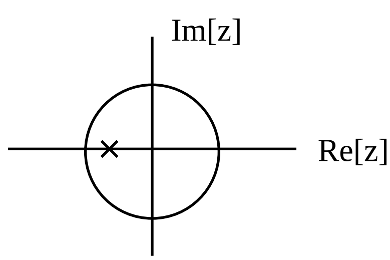
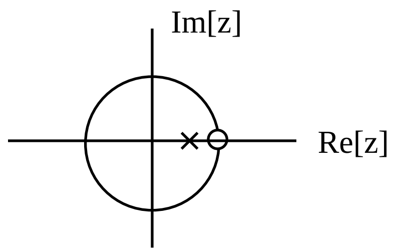


c.



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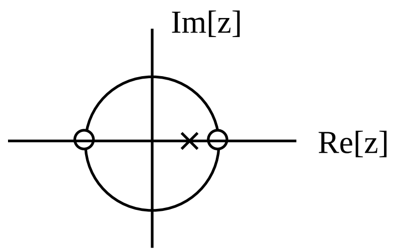
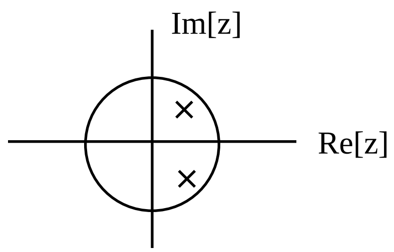
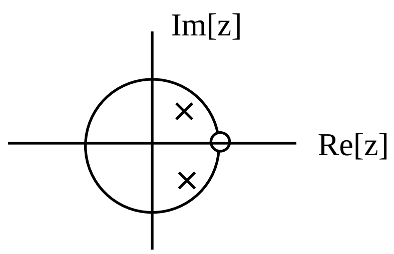


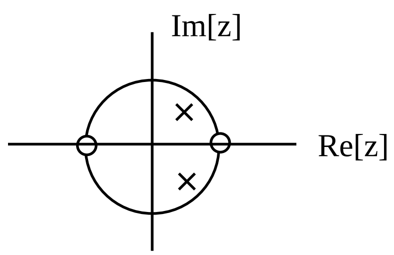
e.



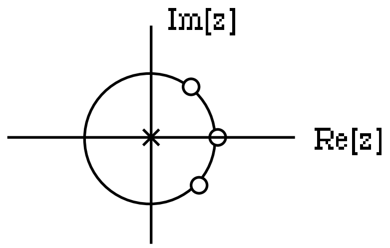
f.



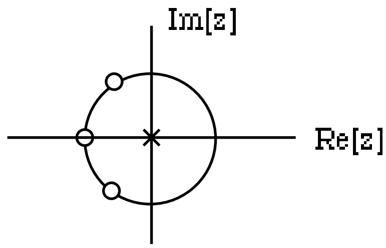
g.



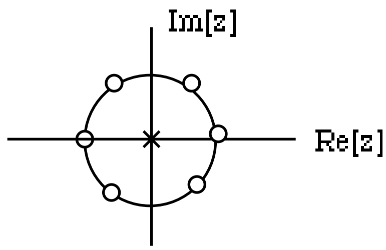
h.



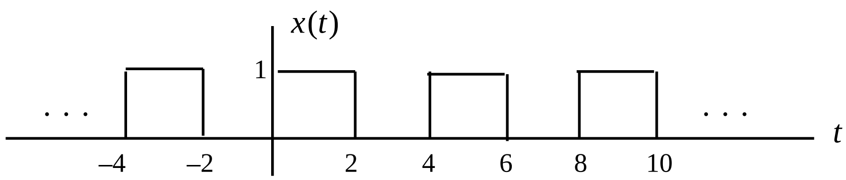
i.



j.

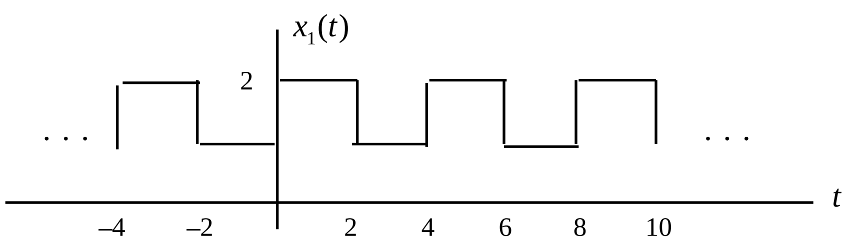


11. Math Review: Given the following periodic signal $x(t)$

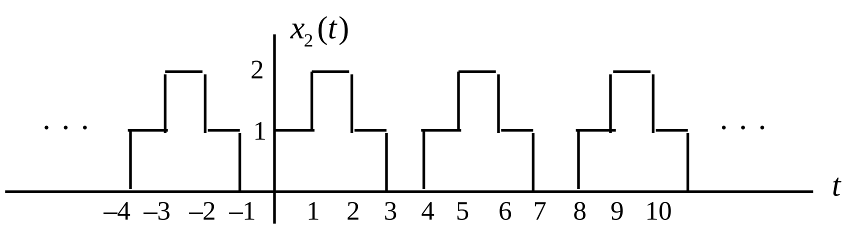


Express each of the following signals in terms of $x(t)$

a.



b.



12. MATLAB - Copy and then save the following M-file as xy_intercept.m

```
function  [x0, y0] = xy_intercept(m,b)
% xy_intercept finds the value of x when y=0 and the value of y when x=0
% for the line  y = mx + b
x0 = -b/m;
y0 = b;
end   % function xy_intercept
```

And then run it in the Command Window with [x0, y0] = xy_intercept(2, -3)

Describe the syntax for how to get the value of more than one variable from the function