ECE 306 - FREQUENCY RESPONSE - INVESTIGATION 10
FREQUENCY RESPONSES OF DISCRETE SYSTEMS - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

We showed in the last Investigation how to find sinusoidal steady state responses of difference equations by expressing the sinusoids as the real parts of complex exponentials as follows

$$y[n] = B\cos\left(2\pi fnT_s + \phi\right) = \mathrm{Re}\left[Be^{j\phi}e^{j2\pi fnT_s}\right]$$

The objective of this Investigation is to find the **frequency responses** of linear time-invariant difference equations - to find how the amplitudes and phases of sinusoidal steady state responses vary as a function of frequency.

1. We begin with a review problem. Given the following difference equation

$$y[n] = 0.6y[n-1] + 2x[n]$$

   a. Find the steady state response to $x(t) = 5\cos(2\pi 500t + 1.2)$ sampled at $f_s = 2500$ samples/sec.
   b. Now find the steady state response to $x(t) = 5\cos(2\pi 1000t + 1.2)$ sampled at $f_s = 2500$ samples/sec
   c. How did increasing the frequency of the sinusoid affect the amplitude and phase of the sinusoidal steady state response

2. In Problem (1) we saw that the amplitude and phase of the sinusoidal steady state response changed as we changed the frequency of the input. The objective of this problem is to obtain a general expression for how the amplitude B and ϕ phase of the sinusoidal steady state response as follows

$$y[n] = B\cos\left(2\pi fnT_s + \phi\right) = \mathrm{Re}\left[Be^{j\phi}e^{j2\pi fnT_s}\right]$$

   varies as a function of frequency f and sampling rate f_s. Suppose in particular that

$$y[n] = 0.6y[n-1] + 2x[n]$$

   with

$$x[n] = A\cos\left(2\pi fnT_s + \theta\right) = \mathrm{Re}\left[Ae^{j\theta}e^{jn2\pi fnT_s}\right]$$

   a. Substitute $x[n] = \mathrm{Re}\left[Ae^{j\theta}e^{jn2\pi fnT_s}\right]$ and $y[n] = \mathrm{Re}\left[Be^{j\phi}e^{jn2\pi fnT_s}\right]$ into the difference equation and solve for $Be^{j\phi}$ as a function of f and $f_s = 1/T_s$
   b. Make use of your result in part (a) to verify your results in Problems (1a) and (1b)

3. From Problem (2) we know that if

$$x[n] = A\cos\left(2\pi fnT_s + \theta\right) = \mathrm{Re}\left[Ae^{j\theta}e^{j2\pi fnT_s}\right] = \mathrm{Re}\left[Xe^{j2\pi fnT_s}\right] \qquad \text{where} \quad X = Ae^{j\theta}$$

is the input to a linear difference equation then the amplitude and phase of the sinusoidal steady state response depend on f and f_s according to expressions like the following

$$Be^{j\phi} = \frac{e^{j2\pi fT_s}}{1+2e^{j2\pi fT_s}}\, X = H\left(e^{j2\pi fT_s}\right)X$$

where $H\left(e^{j2\pi fT_s}\right) = \dfrac{e^{j2\pi fT_s}}{1+2e^{j2\pi fT_s}}$ is the coefficient of $X = Ae^{j\theta}$. We then usually write $Be^{j\phi}$ as $Y\left(e^{j2\pi fT_s}\right)$ and so we have

$$Be^{j\phi} = Y\left(e^{j2\pi fT_s}\right) = H\left(e^{j2\pi fT_s}\right)X$$

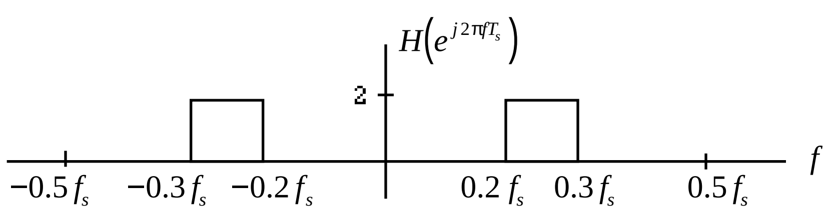
Note in particular that we call $H\left(e^{j2\pi fT_s}\right)$ the **transfer function** of the difference equation. **Memorize** this definition. Then

a. Find the transfer function $H\left(e^{j2\pi fT_s}\right)$ for the difference equation in Problem (2)
b. Calculate your transfer function $H\left(e^{j2\pi fT_s}\right)$ in part (a) for $x(t) = 5\cos(2\pi 1000t + 1.2)$ sampled at $f_s = 2500$ samples/sec
c. Find the sinusoidal steady state response $y[n]$ of a linear difference equation with transfer function as follows

$$H\left(e^{j2\pi fT_s}\right) = \frac{e^{j2\pi fT_s}}{1+2e^{j2\pi fT_s}}$$

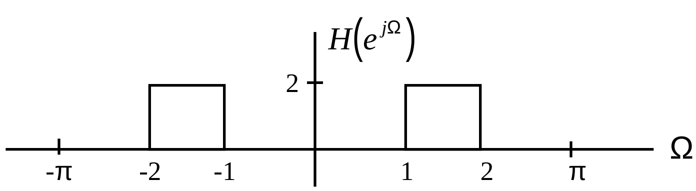
to the input $x(t) = 5\cos(2\pi 1000t + 1.2)$ sampled at $f_s = 2500$ samples/sec. Hint - first find $H\left(e^{j2\pi fT_s}\right)$ and then use it to calculate $Be^{j\phi} = Y\left(e^{j2\pi fT_s}\right) = H\left(e^{j2\pi fT_s}\right)X$
d. How does increasing f_s affect the amplitude of the steady state response in part (c)

4. Given the following ideal frequency response



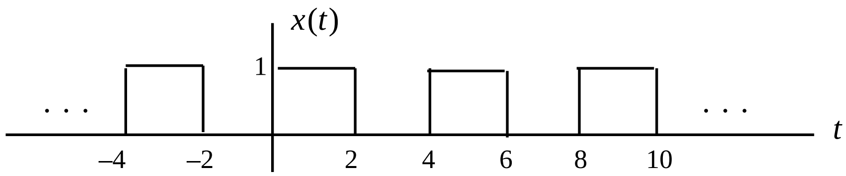
Find the sinusoidal steady state response to $x(t) = 5\cos(2\pi 700t + 1.2)$ if $f_s = 3000$ samples/sec

5. Frequency responses $H\left(e^{j2\pi fT_s}\right)$ like in Problem (4) are often written and plotted as a function of the **normalized frequency** $\Omega = 2\pi fT_s$ as in the following example



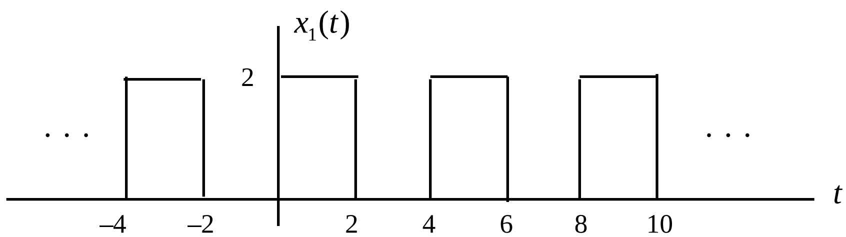
a. What is the steady state response to $x(t) = 3\cos(2\pi 100t)$ if f_s = 400 samples/sec
b. What is the steady state response to $x(t) = 3\cos(2\pi 100t)$ if f_s = 250 samples/sec

6. Math Review: Given the following periodic signal $x(t)$

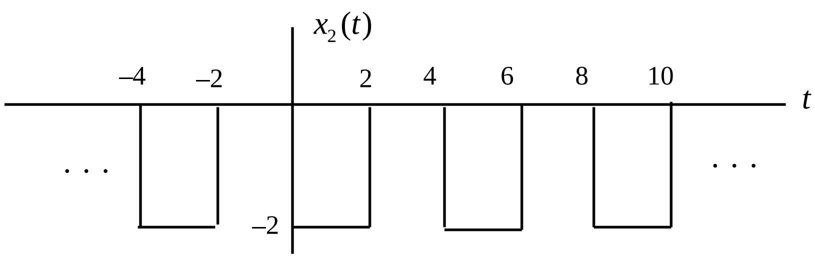


Express each of the following signals in terms of $x(t)$

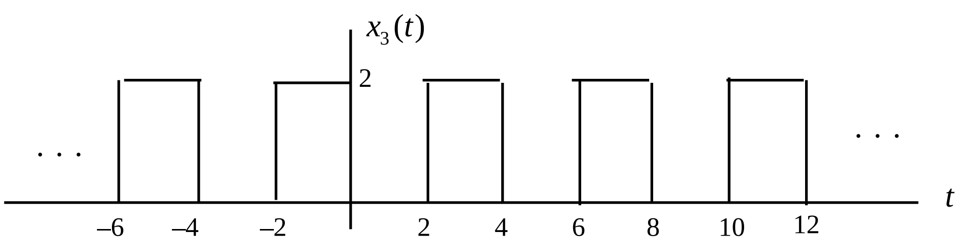
a.



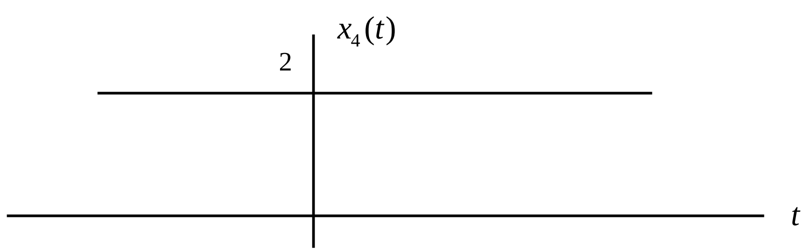
b.



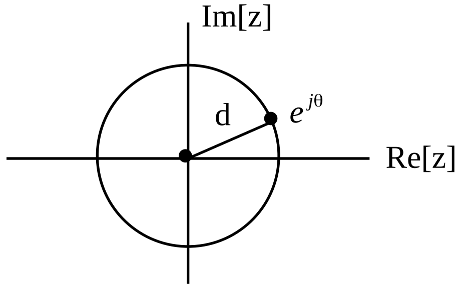
c.



d.



7. Match Review: Sketch the distance d between the two points in the complex plane as follows as a function of θ for $0 \le \theta \le 2\pi$



8. MATLAB - Write and then run an M-file to calculate the response of the following difference equation

$$y[n] = 0.4y[n-2] + 0.9y[n-1] + 1.2x[n] \qquad y[-1] = 0.3$$

for $n = 0$ to $n = 10$

9. MATLAB - Run the following code in the Command Window

```
>>  z = 1 + 2j
>>  x = real(z)
>>  y = imag(z)
>>  r = abs(z)
>>  theta = angle(z)
>>  a = 2 + jcos(2)
>>  b = 1 + j*cos(2)
```

Make use of your results to

a. Explain what j is
b. Sketch z in the complex plane
c. Explain the instruction *real*
d. Explain the instruction *imag*
e. Explain the instruction *abs*
f. Explain the instruction *angle*
g. Explain why a = 2 + jcos(2) gave an error message but z = 2 + j2 did not