ECE 209 - FOURIER SERIES - INVESTIGATION 21
CALCULATION OF FOURIER SERIES

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

We saw in the last investigation that if we add up a sum of sinusoids with frequencies that are **integer multiples** of f_o as follows

$$0,\ f_o,\ 2f_o,\ 3f_o,\ \ldots$$

then the result will be a periodic signal of frequency f_o. The objective of this investigation is to go in the opposite direction and find the sinusoids of frequency $0, f_o, 2f_o, 3f_o, \ldots$ as follows

$$x(t) = c_o + c_1 \cos(2\pi f_o t + \theta_1) + c_2 \cos(2\pi 2 f_o t + \theta_2) + \ldots$$

that add up to a **given** periodic signal x(t). Note that we refer to the frequency $f_o = 1/T$ of a periodic signal as its **fundamental frequency** or **first harmonic**. We then refer to $2f_o$ as the second harmonic, $3f_o$ as the third harmonic and so on.

1. The scheme that **Fourier** came up with for calculating the constant c_o (the dc offset), the coefficients $c_1, c_2, \ldots$ and the phases $\theta_1, \theta_2, \ldots$ of the sinusoids that add up to a given periodic signal x(t) is based on a series of mathematical observations and tricks. Our plan is to illustrate his scheme with the following periodic signal x(t)

$$x(t) = c_o + c_1 \cos(2\pi 100t + \theta_1)$$

of period $T = 0.01$ sec that is simply a sinusoid with a DC offset

   a. The first step is to make use of the trig identity $\cos(x + y) = \cos x \cos y - \sin x \sin y$ to show that $x(t) = c_o + c_1 \cos(2\pi 100t + \theta_1)$ can be expressed in the form

$$x(t) = a_o + a_1 \cos(2\pi 100t) + b_1 \sin(2\pi 100t)$$

   Carry out the trig and find a_o, a_1 and b_1 in terms of c_o, c_1 and θ_1.

   b. Our goal now is to come up with a scheme for finding a_o, a_1 and b_1 in terms of x(t). We will then go back later and show how to obtain the values of c_o, c_1 and θ_1. We begin with a_o. Find an expression for a_o in terms of x(t) by simply integrating both sides of our equation

$$x(t) = a_o + a_1 \cos(2\pi 100t) + b_1 \sin(2\pi 100t)$$

   over a period T and then solve for a_o

   c. Now make use of the following tricks and identities to find a_1 as an integral of x(t)

      (1) Multiply both sides of the equation for x(t) by $\cos(2\pi 100t)$

      (2) Then make use of the trig identities

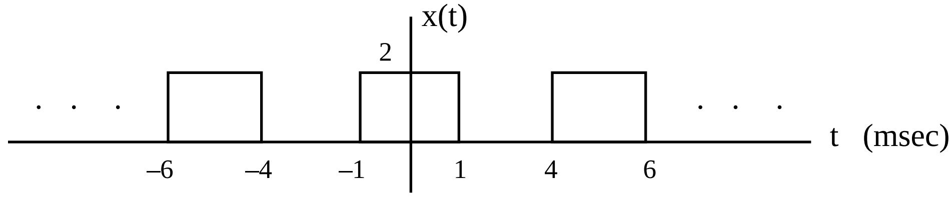
$$\cos^2(x) = 1/2 + 1/2 \cos(2x)$$
$$\cos x \cos y = 1/2 \cos(x + y) + 1/2 \cos(x - y)$$
$$\cos x \sin y = 1/2 \sin(x + y) + 1/2 \sin(x - y)$$

to get an expression for the right hand side of your equation as a constant plus a sum of sinusoids

(3) And finally integrate both sides of your equation over a period T and solve for a_1

d. Go through an analogous set of steps to obtain an expression for b_1 as an integral of x(t)

2. Generalizing on the result of Problem (1) we have that if x(t) is a periodic signal like the following pulse train



then it can be written as a sum of sinusoids as follows

$$x(t) = c_o + c_1 \cos(2\pi f_o t + \theta_1) + c_2 \cos(2\pi 2 f_o t + \theta_2) + \cdots$$

$$= a_o + a_1 \cos(2\pi f_o t) + b_1 \sin(2\pi f_o t) + a_2 \cos(2\pi 2 f_o t) + b_2 \sin(2\pi 2 f_o t) + \cdots + a_k \cos(2\pi k f_o t) + b_k \sin(2\pi k f_o t) + \cdots$$

with

$$a_0 = c_o = \frac{1}{T}\int_T x(t)dt$$

$$a_k = \frac{2}{T}\int_T x(t)\cos(2\pi k f_o t)dt \qquad b_k = \frac{2}{T}\int_T x(t)\sin(2\pi k f_o t)dt$$

Now make use of these integrals to

a. Find a_o for x(t)

b. Find a_1, b_1, a_2, b_2, a_3, b_3 for x(t)

c. Obtain general integral expressions for a_k and b_k of x(t)

3. The objective of this problem is to obtain the c_k's and θ_k's in the Fourier Series of Problem (1) in terms of the a_k's and b_k's we've been working with. Starting from

$$c_k \cos(2\pi k f_o t + \theta_k) = c_k \cos(\theta_k)\cos(2\pi k f_o t) - c_k \sin(\theta_k)\sin(2\pi k f_o t)$$

$$= a_k \cos(2\pi k f_o t) + b_k \sin(2\pi k f_o t)$$

a. Verify that

$$c_k = \sqrt{a_k^2 + b_k^2} \qquad \text{and} \qquad \theta_k = \text{angle}\left(\frac{-b_k}{a_k}\right)$$

b. Then calculate c_o, c_1, θ_1, c_2, θ_2, c_3, θ_3 for the pulse train of Problem (2). Be careful: $a_k < 0$ and $b_k = 0$ imply that $\theta = \pi$

c. Now use Mathcad or a plotting calculator to graph

$$v(t) \approx c_o + \sum_{k=1}^{3} c_k \cos(2\pi k f_o t + \theta_k)$$

Your graph should be pretty recognizable as a pulse train.