ECE 209 - COMPLEX EXPONENTIALS - INVESTIGATION 2
BASIC PROPERTIES - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

The objective of this investigation is to review the properties of complex numbers that we need to simplify the solving of differential equations for sinusoidal steady state responses. Note that from **this time on** some of our equations will contain **j's but no t's;** some will contain **t's but no j's**; and some will contain **both t's and j's**. The **sooner** you get this all sorted out the **better your chances** of doing well in this class.

1. Describe in words the following expressions for multiplying and dividing complex exponentials

$$r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} \qquad \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

   What, in particular, are the magnitudes and phases of the products and quotients of complex exponentials. How would you make use of Euler's Relation to verify these relationships for complex exponentials (it's not necessary to actually do the derivations)

2. Express each of the following as complex exponentials. Be sure to remember that all phase angles are in radians

   a. $z = \left(5e^{j1.2}\right)\left(2e^{-j0.4}\right)$ c. $z = \dfrac{5e^{j1.2}}{1 + j3}$

   b. $z = \dfrac{5e^{j1.2}}{2e^{-j0.4}}$ d. $z = \dfrac{2 + j}{1 + j3}$

3. Make use of Euler's Relation to prove the very important result for complex exponentials that

$$\frac{de^{jbt}}{dt} = jbe^{jbt}$$

   How is this result analogous to the corresponding result for real exponentials

4. Make use of Euler's Relation to verify the very, very important result that sinusoids can be expressed in terms of complex exponentials as follows

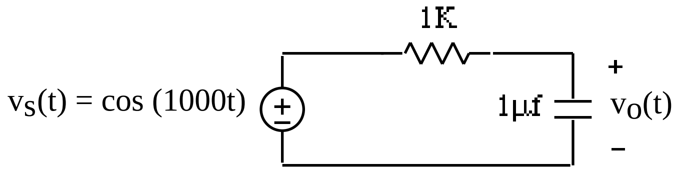
$$a.\ v(t) = \frac{Ae^{j\theta}}{2} e^{j1000t} + \frac{Ae^{-j\theta}}{2} e^{-j1000t} = A\cos(1000t + \theta)$$

$$b.\ v(t) = \mathrm{Re}\left[Ae^{j\theta} e^{j1000t}\right] = A\cos(1000t + \theta)$$

5. Express v(t) = 7.5 cos (2000t + 1.1) as the real part of a complex exponential as in Problem (4b) above. This is how we're going to be expressing our sinusoids in this class

6. The objective of this problem is to illustrate how complex exponentials can be used to simplify

the calculation of sinusoidal steady responses of a circuit like the following



with differential equation

$$v_o' + 10^3 v_o = 10^3 \cos(10^3 t)$$

As usual, the strategy is to substitute $v_f(t) = B\cos(1000t + \varphi)$ into the differential equation for the forced response

$$v_f' + 10^3 v_f = 10^3 \cos(10^3 t)$$

and then solve for B and φ. But now we express the sinusoids $v_S(t)$ and $v_f(t)$ in terms of the real parts of complex exponentials as follows

$$v_s(t) = \cos(10^3 t) = \mathrm{Re}\left[e^{j1000t}\right]$$

and

$$v_f(t) = B\cos(10^3 t + \phi) = \mathrm{Re}\left[Be^{j\phi}e^{j1000t}\right]$$

to obtain

$$\frac{d}{dt}\,\mathrm{Re}\left[Be^{j\phi}e^{j1000t}\right] + 10^3\,\mathrm{Re}\left[Be^{j\phi}e^{j1000t}\right] = 10^3\,\mathrm{Re}\left[e^{j1000t}\right]$$

So far so good. But in order to be able to calculate B and φ we need to make use of the following relationships

(1) $\mathrm{Re}\,[z_1] + \mathrm{Re}\,[z_2] = \mathrm{Re}\,[z_1 + z_2]$

(2) $K\,\mathrm{Re}\,[z] = \mathrm{Re}\,[Kz]$ (when K is real)

(3) $\frac{d}{dt}\,\mathrm{Re}\left[e^{jbt}\right] = \mathrm{Re}\left[\frac{d}{dt}e^{jbt}\right] = \mathrm{Re}\left[jbe^{jbt}\right]$

(4) If $\mathrm{Re}\left[Ae^{j\theta}e^{jbt}\right] = \mathrm{Re}\left[Be^{j\phi}e^{jbt}\right]$ for all time t then $Ae^{j\theta} = Be^{j\varphi}$

a. Find an example to illustrate Relationship (1)
b. Find an example to illustrate Relationship (2).
c. Prove Relationship (3)
d. Prove Relationship (4). Hint - express $\mathrm{Re}\left[Ae^{j\theta}e^{jbt}\right]$ and $\mathrm{Re}\left[Be^{j\phi}e^{jbt}\right]$ as cosines
e. Make use of the Relationships (1)-(4) to express each of the following voltages v(t) in the form $\mathrm{Re}[Ae^{j100t}]$ where A is a complex exponential

   (i) $v(t) = \mathrm{Re}\,[j2\,e^{j100t}] + \mathrm{Re}\,[2e^{j100t}]$

   (ii) $v(t) = \mathrm{Re}\,[j2\,e^{j100t}] + 2\,\mathrm{Re}\,[2e^{j100t}]$

(iii) $v(t) = \frac{d}{dt}\,\mathrm{Re}\left[2e^{j100t}\right] + 200\mathrm{Re}\left[2e^{j100t}\right]$

f. Make use of the Relationships (1)-(4) to find B and φ in

$$\mathrm{Re}\left[Be^{j\varphi}e^{j100t}\right] = \mathrm{Re}\left[(1 + j2)\,e^{j100t}\right]$$

g. Make use of the Relationships (1)-(4) to find B and φ in our differential equation

$$\frac{d}{dt}\,\mathrm{Re}\left[Be^{j\phi}e^{j1000t}\right] + 10^3\,\mathrm{Re}\left[Be^{j\phi}e^{j1000t}\right] = 10^3\,\mathrm{Re}\left[e^{j1000t}\right]$$

7. Note that we refer to $Be^{j\varphi}$ - the coefficient of e^{j1000t} in the following expression

$$v_o(t) = B\cos(1000t + \varphi) = \mathrm{Re}\left[Be^{j\varphi}e^{j1000t}\right]$$

as the **phasor** corresponding to the sinusoid. Note also that since the values of B and φ in general depend on the frequency ω of the sinusoid we usually write $v_o(t)$ as follows

$$v_o(t) = B\cos(\omega t + \varphi) = \mathrm{Re}\left[Be^{j\varphi}e^{j\omega t}\right] = \mathrm{Re}\left[V_o(j\omega)\,e^{j\omega t}\right]$$

with $V_o(j\omega) = Be^{j\varphi}$ equal to the phasor of the voltage $v_o(t)$

a. Express $v_o(t) = 5\cos(1000t + \pi/4)$ as the real part of a complex exponential. Then find the corresponding phasor $V_o(j1000)$

b. Express $v_o(t)$ as a sinusoid if its phasor is $V_o(j1000) = 2e^{-j1.3}$

Note that we use **capitals** like V(j1000) and I(j1000) for phasors in contrast to small letters for time signals like v(t) and i(t)

8. Make up an example to illustrate that $\mathrm{Re}\left[z_1 z_2\right] \neq \mathrm{Re}\left[z_1\right]\mathrm{Re}\left[z_2\right]$. **Memorize** this result. It's a common mistake to think these two expressions are equal