ECE 209 - COMPLEX EXPONENTIALS - INVESTIGATION 1
BASIC PROPERTIES - PART I

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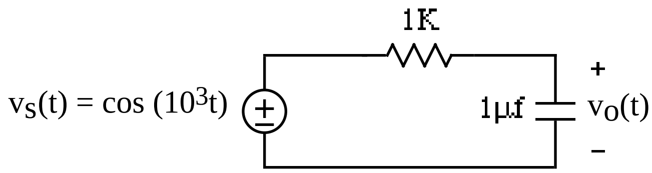
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

Our main goal in ECE 209 is to develop efficient methods for calculating how the amplitudes and phases of sinusoidal steady state responses of linear circuits vary as a function of frequency. We begin with a brief review of how we calculated sinusoidal steady state responses in ECE 207 and then show how using complex numbers makes the analysis easier.

We know from ECE 207 that we can find the sinusoidal steady state response - the sinusoid that's left after the transient dies out - of a **linear** circuit like the following



by first substituting the forced response $v_f(t)$ as follows

$$v_f(t) = B\cos(1000t + \varphi)$$

into the circuit's differential equation

$$v_f' + 10^3 v_f = 10^3\cos(1000t)$$

to obtain

$$\frac{d}{dt}\big(B\cos(1000t+\varphi)\big) + 10^3\big(B\cos(1000t+\varphi)\big) = 10^3\cos(1000t)$$

and then making use of trig identities to solve for B and φ. As we will see this analysis can be done much easier with the aid of complex exponentials of the form e^{jx} where

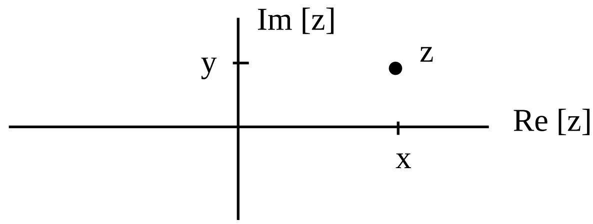
$$j \equiv \sqrt{-1}$$

The objective of this investigation is to review complex numbers and the basic properties of complex exponentials.

1. Given $z_1 = 1 + j2$ and $z_2 = 3 - j$ find
   a. $z = z_1 + z_2$
   b. $z = z_1 z_2$
   c. $z = z_1^*$ = complex conjugate of z_1
   d. $z = z_1 z_1^*$

2. What are the rectangular coordinates x and y of the complex number $z = 2 - j4$.

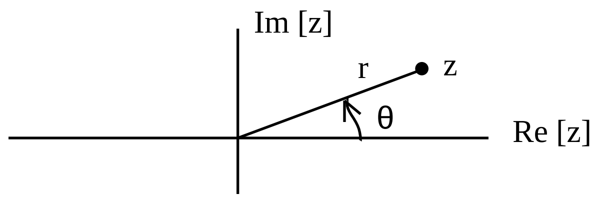
3. Describe in words how the rectangular coordinates x and y of a complex number $z = x + jy$ are related to its location in the complex plane as follows



4. Plot each of the following in the complex plane:

$$z_1 = 1,\ \ z_2 = j,\ \ z_3 = 1 + j \ \text{ and } \ z_4 = -1 - j$$

5. Describe in words how the polar coordinates r and θ of a complex number z are related to its location in the complex plane as follows



Memorize the fact that we refer to

$r = |z|$ as the **magnitude** of z and
$\theta =$ z as the **phase** or **phase angle** of z.

6. Plot each of the following in the complex plane:

z_1 with polar coordinates $r_1 = 2$ and $\theta_1 = \pi/4$
z_2 with polar coordinates $r_2 = 1$ and $\theta_2 = -\pi/4$
z_3 with polar coordinates $r_3 = 1$ and $\theta_3 = \pi$

Note that in general we'll be expressing our angles in **radians**

7. What are the polar coordinates r and θ of $z = 3 + j2$

8. Find and correct the mistake in the following expression

$$|z| = |3 + j5| = \sqrt{3^2 + (j5)^2}$$

9. Find the polar coordinates r and θ of $z = a\cos(b) + j\,a\sin(b)$. **Memorize** this result.

10. Make use of your result in Problem (9) to find the rectangular coordinates of the complex number z with polar coordinates $r = |z| = 2$ and $\theta =$ z = 0.3 radians.

11. Given the two complex numbers
z_1 with polar coordinates $r_1 = |z_1| = 2$, $\theta_1 =$ $z_1 = 0.5$

z_2 with polar coordinates $r_2 = |z_2| = 3$, $\theta_2 = \quad z_2 = -1.2$

a. Make use of the trig identity

$$(a\cos(b) + j\,a\sin(b))\,(c\cos(d) + j\,c\sin(d)) = a\,c\,\cos(b+d) + j\,a\,c\sin(b+d)$$

to find the polar coordinates of $z_1 z_2$

b. Make use of the trig identity

$$\frac{a\cos(b) + ja\sin(b)}{c\cos(d) + jc\sin(d)} = \frac{a}{c}\cos(b-d) + j\frac{a}{c}\sin(b-d)$$

to find the polar coordinates of z_1/z_2

12. Use the atan key on your calculator to calculate the phase of $z = -1 - j$ and then use the rectangular to polar conversion key or its equivalent. Which "answer" is right. Why

13. Calculate

$$z(t) = \frac{d}{dt}\left(\cos(100t) + j\sin(100t + \pi/4)\right)$$

14. The objective of this and the rest of the problems in this investigation is to introduce complex exponentials. We start with a look at real exponentials since it was one of the main goals of Euler and his contemporaries to **define** complex exponentials - to specify their real and imaginary parts - so they would have the same terrific properties that real exponentials do. Describe the really nice "rules" for multiplying, dividing and differentiating real exponentials of the form c^{ax} that Euler wanted to preserve for complex exponentials.

15. Now, Euler's genius was that he not only figured out a way to **define** complex exponentials $z = c^{jb}$ so they would have the same terrific properties as real exponentials but also in such a way that we get **Euler's Relation** as follows

$$e^{jb} = \cos(b) + j\sin(b)$$

for the special case of $c = e$. Euler accomplished this fete by taking the Taylor's Series Expansion for real exponentials c^x as follows

$$c^x = 1 + (\ln c)\,x + \frac{(\ln c)^2}{2!}\,x^2 + \frac{(\ln c)^3}{3!}\,x^3 + \cdots$$

and making the substitution $x = jb$ to obtain his definition for complex exponentials as follows

$$c^{jb} \equiv 1 + (\ln c)(jb) + \frac{(\ln c)^2}{2!}(jb)^2 + \frac{(\ln c)^3}{3!}(jb)^3 + \cdots$$

$$= 1 + j(\ln c)b - \frac{(\ln c)^2}{2!}\,b^2 - j\,\frac{(\ln c)^3}{3!}\,b^3 + \cdots$$

With $c = e$ we then have

$$e^{jb} = 1 + j(\ln e)b - \frac{(\ln e)^2}{2!} b^2 - j \frac{(\ln e)^3}{3!} b^3 + \cdots$$

$$e^{jb} = \left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \cdots\right) + j\left(b - \frac{b^3}{3!} + \cdots\right)$$

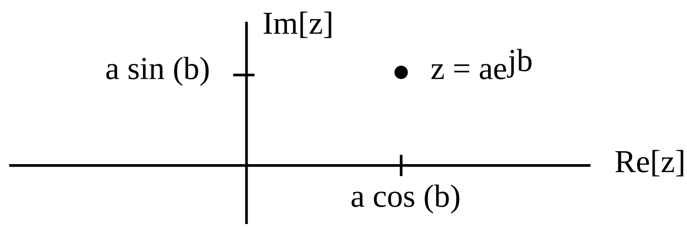
which gives us Euler's Relation

$$e^{jb} = \cos(b) + j\sin(b)$$

Therefore we have, by Euler's Relation, that

$$z = a e^{jb} = a\cos(b) + ja\sin(b)$$

is the complex number with rectangular coordinates $x = a\cos(b)$ and $y = a\sin(b)$ located in the complex plane as follows



Verify that z's polar coordinates are $r = |z| = a$ and $\theta = \angle z = b$ and so

$$z = ae^{jb} = |z|e^{j\angle z}$$

Euler's Relation turns out to be particularly fortuitous because it enables us to express sinusoids in terms of complex exponentials. Which enables us to take advantage of their really nice properties to more easily calculate sinusoidal steady state responses. **Memorize** the results of this problem.

16. Use your results from above to find the rectangular and polar coordinates of $z = 3e^{j2}$. Plot z in the complex plane.

17. Express $z = 2 + j2$ as a complex exponential. What are its polar coordinates. Plot z in the complex plane.