ECE 130 - INDUCTION AND RECURSION - INVESTIGATION 9
SOLVING RECURSIVE EQUATIONS

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include appropriate graphs and tables.

In the last Investigation we showed how Fibonacci and other sequences can be defined recursively. The main goal of this Investigation is to show how to use back up subsitution to find closed form solutions of recursive equations. And to then introduce the Tower of Hanoi problem.

1. Given the following recursively defined equation

$$S(n) = S(n-1) + 2n \qquad S(0) = 0$$

   a. Set up a Table for S(n) for n=1 to n=5
   b. What is S(n) the sum of

2. From Problem (1) we know that if S(n) is equal to the following recursive equation

$$S(n) = S(n-1) + 2n \qquad S(0) = 0$$

   then $S(n)$ is the sum of the first n even integers. Now a computer can obtain S(n) for any n we are ever to likely to need in a real application. It's often valuable, however, to have a nice closed form equation for S(n) so we can *see* what's going on as *n* increases. There are several ways to obtain such solutions. Suppose we want to find $S(5)$. Then

$$S(5) = S(4) + 2\ \ 5 \quad \rightarrow \quad S(5) = S(3) + 2\ \ 4 + 2\ \ 5 \quad \rightarrow \quad \cdots$$

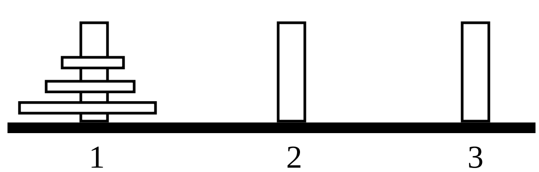
   and so on. We call this *backing up* the recurrence
   a. Continue the backing up until you obtain S(5) as a function of S(0) and a sum of constants
   b. Make use of your result in part (a) to verify that S(5) is the sum of the first 5 even integers
   c. Generalize on your result in part (b) to find $S(n)$ as a sum. Use summation notation
   d. Make use of induction on your result in part (c) to show that $S(n) = n(n+1)$

3. Given the following recursive equation

$$x(n) = 0.5x(n-1) + 2 \qquad x(0) = 0$$

   a. Use backing up to find an expression for x(n) as a function of x(0) and a sum of terms
   b. Make use of your expansion in part (a) to obtain a closed form solution of x(n). Note that the sum is a geometric sum

4. The objective of this problem is to introduce the classic Tower of Hanoi problem as follows



to move the "rings" from peg 1 to peg 2 under the following constraints

(1) Only one ring can be moved at a time
(2) A bigger ring cannot be put on top of a smaller ring

a. We begin with the case of n=2 rings. Draw a sequence of pictures to show how to move the two rings to peg 2 subject to the constraints.
b. Now draw a sequence of drawings that show to move n=3 rings to peg 2. Hint - first move two rings to peg 3
c. Explain how step (b) is using recursion
d. Generalize on your result in part (c) to explain how recursion can be used to move n rings from peg 1 to peg 2

5. The objective of this problem is to count the moves as the rings are moved in the Tower of Hanoi problem
   a. How many moves does it take to move two rings to peg 2
   b. How many moves did it take to move three rings to peg 2
   c. Generalize on the results in parts (a) and (b) to write the recursive equation for the number of moves S(n) it takes to move n rings to peg 2
   d. Find a closed form solution for your S(n) in part (c)