ECE 130 - MATRICES - INVESTIGATION 24
INTRODUCTION TO MATRICES - PART II

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include appropriate graphs and tables.

In the last Investigation we showed how to put linear equations in matrix form and how to solve them in some special simple cases. The objective of this Investigation is to introduce the basic idea of how general matrix equations can be solved.

1. We begin with a review problem
   a. Solve for x_1 and x_2 in the following matrix equation

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

   b. Solve for x_1 and x_2 in the following matrix equation

$$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

2. As we said in the last Investigation putting linear equations in matrix form as follows

$$Ax = b$$

is so nice because it makes a system of linear equations look like just one equation. Now the objective of this and the following problems is to show how we can define matrix addition, subtraction and multiplication so we can solve matrix equations in the "same way" we solve single equations as follows

$$ax = b \qquad x = \frac{1}{a}b \qquad x = a^{-1}b$$

In particular by multiplying b by the inverse of A as follows

$$Ax = b \qquad x = A^{-1}b$$

We begin with **matrix addition** as illustrated in the following example

$$S = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 4 \end{pmatrix}$$

Note that the two matrices being added as well as the result - all have the *same dimension*
   a. Make use of the example to explain how matrix addition is done
   b. Add the following two matrices

$$S = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 3 & -2 \\ 5 & 2 & 3 \end{pmatrix}$$

   c. Do your own example of matrix addition

3. Make use of your definition for matrix addition in Problem (2) to define matrix subtraction

4. Make use of your definition for matrix subtraction to
   a. Subtract the following two matrices

$$D = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 3 & -2 \\ 5 & 2 & 3 \end{pmatrix}$$

   b. Do your own example of matrix subtraction

5. What happens when we subtract a matrix from itself. Note that we call a matrix of all 0's a **null matrix**

6. Now if you've never seen matrix multiplication before it will look kind of contrived. But in fact it's what we've been doing all along when we put linear equations like the following

$$a_{11}x_1 + a_{12}x_2 = b_1$$

in matrix form as follows

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \end{pmatrix}$$

Now the product of the 1×2 row matrix

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix}$$

with the 2×1 column matrix

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

gives us the following sum of products

$$\begin{pmatrix} a_{11} & a_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 \\ + \\ a_{12}x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \end{pmatrix}$$

   a. Make use of this definition of matrix multiplication to show that

$$B = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \end{pmatrix}$$

   b. Calculate the matrix product

$$B = \begin{pmatrix} 2 & 7 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

   c. Make up your own example of matrix multiplication

7. Generalize on the results of Problem (6) to verify that

$$\begin{pmatrix} 3 & 4 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -3 \end{pmatrix}$$

8. Generalizing on the last problem we have that if

$$C = A\ B$$

then

$$c_{jk} = \text{sum of the products of the } j'th \text{ row of } A \text{ and the } k'th \text{ column of } B$$

   a. Make use of this result to find c_{32} in the following product

$$C = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & -1 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 5 \\ -1 & 2 \end{pmatrix}$$

   b. How does the number of columns of A have to be related to the number of rows of B in order for us to be able to calculate the product $C = A\ B$

9. The objective of this problem is to see how reversing the order of two matrices affects the value of their product. Given

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

   a. Calculate $A\ B$
   b. Calculate $B\ A$
   c. How is the relationship between the matrix products $A\ B$ and $B\ A$ different from that between $a\ b$ and $b\ a$ for numbers a and b. **Memorize** this result

10. Given the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

   a. Describe I
   b. Calculate $I\ B$
   c. How is the product $I\ B$ related to B
   d. Now calculate $B\ I$
   e. How is the product $B\ I$ related to B

11. In Problem (10) we saw that multiplying B by the I matrix as follows

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gave us back B just like what happens when we multiply a number by 1. We call such matrices - matrices with 1's along the diagonal and 0's every place else as follows

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

identity matrices. **Memorize** this term. Then

a. Verify that $B\ I = I\ B = B$ when

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

b. Make up your own example to illustrate that I is the identity matrix

12. Now for matrix "division". We say that A^{-1} is the **inverse** of the square matrix A if

$$A\ A^{-1} = A^{-1}\ A = I$$

a. Verify that

$$A^{-1} = \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} \text{ is the inverse of } A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

b. Find the inverse matrix A^{-1} of

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

13. Now that we have inverse matrices we can use them to solve linear equations $Ax = b$. All we have to do is multiply both sides by A^{-1} as follows

$$A^{-1}Ax = A^{-1}b \qquad Ix = A^{-1}b \qquad x = A^{-1}b$$

Make use of this result to solve

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$