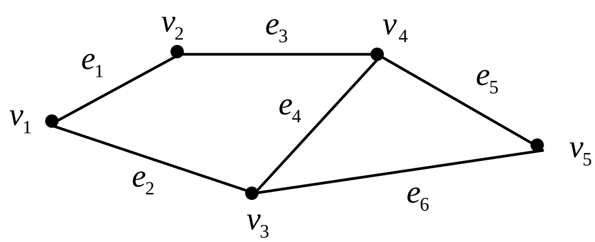
ECE 130 - GRAPHS - INVESTIGATION 17
CONNECTIVITY OF GRAPHS

WINTER 2004 **A.P. FELZER**

To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include appropriate graphs and tables.

In the last Investigation we showed how graphs can be used to represent relations. Another class of important problems involves finding paths in networks like telephone networks and highway systems. Or equivalently finding paths from one vertex to another in corresponding graphs.

1. Two vertices in a graph are connected by a path if there is a sequence of edges from one to the other. Find two paths from v_1 to v_5 in the following undirected graph

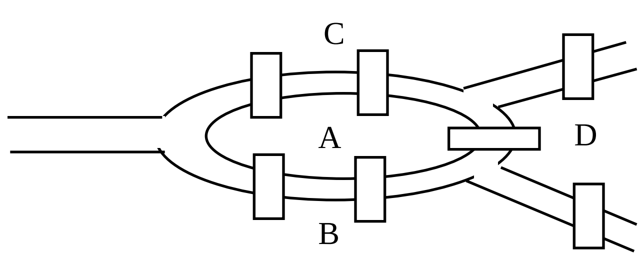


2. A **circuit** is a path that ends at the same vertex at which it starts. Draw a graph and identify a circuit.

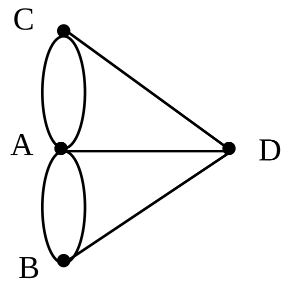
3. A path is **simple** if there is no edge it traverses more than once. Draw a graph and then find a path that is
    a. Simple
    b. Not simple

4. Given a graph G
    a. What does it mean for G to be **connected**
    b. Draw a graph that is connected
    c. Draw a graph that is not connected

5. This and the next three problems is about finding paths in connected graphs that traverse every edge exactly once. This problem originated with the Konigsberg Bridge problem as follows: Is there a path across the Konigsberg Bridges as follows

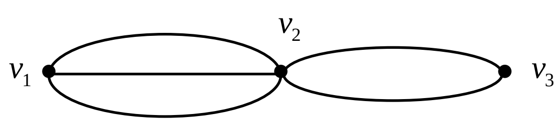


in which every bridge is traversed exactly once. Euler's insight was to represent the bridges as edges on a graph as follows



Make use of Euler's graph to find such a path for the Konigsberg Bridges.

6. From the previous problem we know that the graph for the Konigsberg Bridges has a path that traverses every bridge exactly once. Such paths are called **Euler paths**. See if you can find an Euler path for the following graph



7. From the previous two problems we know that some graphs have Euler paths and some don't. The difference between graphs that have Euler paths and those that don't is the degrees of their vertices
   a. From Investigation 15 we know that the sum of the degrees of all the vertices in a graph must be even as follows

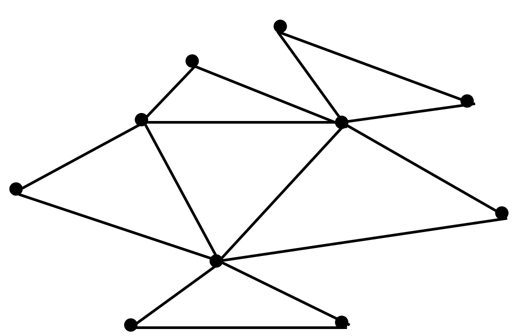
   $$2e = \sum_{v \in V} \deg(v)$$

   Make use of this result to show that if one vertex has an odd degree then so must at least one other
   b. Make use of your result in part (a) to show that if the degree of a vertex in a given graph is odd then that graph can't have an Euler path. Hint - First show that if a graph has a vertex with an odd degree then an Euler path would have to end at it.

8. From Problem (7) we know that if a graph has an Euler path then its degree must be even. Euler showed that if the degree of every vertex of a graph is even then it has an Euler path. The proof is based on the following algorithm -

   (1) Start at an arbitrary vertex and trace out as large a circuit as you can
   (2) Repeat part (a) for each of the remaining connected graphs made from the edges still left
   (3) Repeat part (b) until all edges are in some circuit
   (4) Put together all the circuits to form an Euler path

   Illustrate this algorithm for the following graph



9. This problem is about finding circuits in connected graphs that pass through each vertex exactly once. These are called **Hamilton circuits**. Surprising as it may seem there is no general way to look at an arbitrary graph to determine if it has a Hamilton circuit. Draw a graph that
   a. Has a Hamilton circuit
   b. Doesn't have a Hamilton circuit